# Completely Positive Maps & Stinespring Theorem

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# Recall setup

- Separable complex Hilbert space $H$
- States $\mathcal{S}(H) := \{T \in \mathcal{B}(H) : T \geq 0, \mathrm{tr}(T) = 1\}$
- Observables $\mathcal{A}(H) := \{X \in \mathcal{B}(H) : A = A^*\}$
- Pairing of state and observable: $(T, X) \mapsto \mathrm{tr}(TX)$ (Born's rule)

  e.g. for $T = |\psi\rangle\langle\psi|$ representing a pure state we have

$$\mathrm{tr}(|\psi\rangle\langle\psi|X) = \langle\psi|X|\psi\rangle_H \tag{1}$$

$M : \mathcal{L}_1(H) \to \mathcal{L}_1(H)$ mapping states to states i.e.

(i) linear $M(\alpha S + \beta T) = \alpha M(S) + \beta M(T)$

(ii) preserve positivity $M(S) \geq 0$

(iii) preserve trace $\mathrm{tr}(M(S)) = 1$

for $S, T \in \mathcal{S}(H), \alpha, \beta \in \mathbb{C}$.

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Want $M^* : \mathcal{B}(H) \to \mathcal{B}(H)$ specified by the pairing:

$$\mathrm{tr}(M(T)X) = \mathrm{tr}(TM^*(X)) \ . \tag{2}$$

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This completely specifies the map $M^*$ by [Attal, , Thm. 2.12 (2)].

(i) linear $M^*(\alpha X + \beta Y) = \alpha M^*(X) + \beta M^*(Y)$
(ii) preserve positivity $M^*(X) \geq 0$ if $X \geq 0$
(iii) preserve identity $M^*(\text{id}) = \text{id}$

for $X, Y \in \mathcal{A}(H), \alpha, \beta \in \mathbb{C}$.

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Entanglement.

# Complete Positivity

**Definition**

A linear map $\mathcal{M} : \mathcal{B}(H) \to \mathcal{B}(H)$ is called **k- positive** if

$$\mathcal{M} \otimes \mathsf{id}_k : \mathcal{B}(H) \otimes \mathbb{C}^{k\times k} \to \mathcal{B}(H) \otimes \mathbb{C}^{k\times k} \tag{3}$$

is positive. $\mathcal{M}$ is called **completely positive** if it is $k$-positive for every $k \in \mathbb{N}$.

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**Question**

Why only for finite dimensional auxiliary Hilbert spaces and not for arbitrary? Is "completely positive" as given here strictly weaker?
[or1426 (https://mathoverflow.net/users/130032/or1426), ] says the two notions are equivalent.

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# Not every positive map is completely positive

Consider $\mathcal{M} : \mathbb{C}^{2\times 2} \to \mathbb{C}^{2\times 2}$ via conjugation i.e. $\mathcal{M}(A) = A^*$. Then

$$\mathcal{M}(A^*A) = (A^*A)^* = A^*A \geq 0 \ , \tag{4}$$

so $\mathcal{M}$ is positive,

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but the map $\mathcal{M} \otimes \mathrm{id}_2 : \underbrace{\mathbb{C}^{2\times 2} \otimes \mathbb{C}^{2\times 2}}_{\simeq \mathbb{C}^{4\times 4}} \to \underbrace{\mathbb{C}^{2\times 2} \otimes \mathbb{C}^{2\times 2}}_{\simeq \mathbb{C}^{4\times 4}}$ maps

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$

where the first matrix has spectrum $\{0, 2\}$, but the latter has $\{-1, 1\}$.

# Not every positive map is completely positive

(This map is not linear, but anti-linear. But there are other examples. See [(https://mathoverflow.net/users/56920/arnold neumaier), ] for maps that are $k$-positive but not $(k+1)$-positive.)

# Stinespring Representation Theorem [Attal, , Thm. 6.19]

**Theorem (Stinespring)**

*Let $\mathcal{M} : \mathcal{B}(H) \to \mathcal{B}(H)$ be a completely positive map. Then there exists*

- *a Hilbert space $K$,*
- *a $*$-representation $\pi : \mathcal{B}(H) \to \mathcal{B}(K)$, and*
- *bounded linear operator $V : H \to K$ s.t.*

$$\mathcal{M}(A) = V^* \pi(A) V, \quad A \in \mathcal{B}(H). \tag{6}$$

*Conversely, every map of the above form is completely positive.*

# Consequences of Stinespring

**Corollary**

*Unitary transformations* $A \mapsto U^* A U$ *are completely positive.*

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$^*$*-homomorphisms* $\pi : \mathcal{B}(H) \to \mathcal{B}(H)$ *are completely positive.*

# Converse Direction

Let $\mathcal{M}(\cdot) = V^* \pi(\cdot) V$ as above, $\hat{A} = \hat{B}^* \hat{B} \in \mathcal{B}(H \otimes \mathbb{C}^n)$ be positive. Write $\mathcal{M}_n = \mathcal{M} \otimes \mathrm{id}_n$.

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We want to show that for any $\hat{v} \in H \otimes \mathbb{C}^n$

$$\left\langle \hat{v}, \mathcal{M}_n(\hat{A})\hat{v} \right\rangle_{H\otimes\mathbb{C}^n} \geq 0. \tag{7}$$

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Choose an ONB $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$. Then

$$\hat{B} = \sum_{i,j=1}^{n} B^i_j \otimes |e_j\rangle\langle e_i|, \qquad \hat{v} = \sum_{i=1}^{n} v_i \otimes e_i \quad . \tag{8}$$

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$$\left\langle \hat{v}, \mathcal{M}_n(\hat{A})\hat{v} \right\rangle_{H\otimes\mathbb{C}^n}$$

$$= \sum_{i',j',i,j,k=1}^{n} \underbrace{\left\langle v_{i'} \otimes e_{i'}, \left(\mathcal{M}(B_k^{i\,*} B_k^{j}) \otimes \left|e_i\right\rangle \left\langle e_j\right|\right)(v_{j'} \otimes e_{j'}) \right\rangle_{H\otimes\mathbb{C}^n}}_{\left\langle v_{i'}, \left(\mathcal{M}(B_k^{i\,*} B_k^{j})\right) v_{j'} \right\rangle_H \cdot \underbrace{\langle e_{i'}, |e_i\rangle \left\langle e_j|e_{j'}\right\rangle_{\mathbb{C}^n}}_{=\delta_{i,j}^{i',j'}}}$$

# Converse Direction

$$
\begin{aligned}
&\left\langle \hat{v}, \mathcal{M}_n(\hat{A})\hat{v} \right\rangle_{H\otimes\mathbb{C}^n} \\
&= \sum_{i',j',i,j,k=1}^{n} \underbrace{\left\langle v_{i'} \otimes e_{i'}, \left(\mathcal{M}(B_k^{i\,*} B_k^j) \otimes \left|e_i\right\rangle \left\langle e_j\right|\right)(v_{j'} \otimes e_{j'}) \right\rangle_{H\otimes\mathbb{C}^n}}_{\left\langle v_{i'}, \left(\mathcal{M}(B_k^{i\,*} B_k^j)\right) v_{j'} \right\rangle_H \cdot \underbrace{\left\langle e_{i'}, \left|e_i\right\rangle \left\langle e_j\right| e_{j'} \right\rangle_{\mathbb{C}^n}}_{=\delta_{i,j}^{i',j'}}} \\
&= \sum_{i,j,k=1}^{n} \left\langle v_i, \mathcal{M}(B_k^{i\,*} B_k^j) v_j \right\rangle_H
\end{aligned}
$$

# Converse Direction

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$$= \sum_{i,j,k=1}^{n} \left\langle v_i, \mathcal{M}(B_k^{i\,*} B_k^j) v_j \right\rangle_H = \sum_{i,j,k=1}^{n} \left\langle V v_i, \pi(B_k^{i\,*} B_k^j) V v_j \right\rangle_H$$

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$$= \sum_{k=1}^{n} \left\langle \sum_{i=1}^{n} \pi(B_k^i) V v_i, \sum_{j=1}^{n} \pi(B_k^j) V v_j \right\rangle_H$$

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$$= \sum_{k=1}^{n} \left\| \sum_{i=1}^{n} \pi(B_k^i) V v_i \right\|_H^2 \geq 0$$

# Bibliography


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URL:https://mathoverflow.net/q/327791 (version: 2019-04-11).

Strategy: $K := \overline{\left( \mathcal{B}(\mathcal{H}) \bar{\otimes} \mathcal{H} / \ker(\langle\cdot,\cdot\rangle) \right)}^{\langle\cdot,\cdot\rangle}$

Consider the alg. tensor product $\mathcal{B}(H) \bar{\otimes} H = \operatorname{span}\{ A \otimes v : A \in \mathcal{B}(\mathcal{H}), v \in \mathcal{H} \}$

For $\sum_{i=1}^n A_i \otimes x_i$, $\sum_{i=1}^n B_i \otimes y_i \in \mathcal{B}(\mathcal{H}) \bar{\otimes} \mathcal{H}$ define

$$\hat{A} = \sum_{i=1}^n A_i \otimes |e_i\rangle\langle e_1| \, , \quad \hat{B} = \sum_{j=1}^n B_j \otimes |e_j\rangle\langle e_1| \quad \in \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)$$

and $\hat{x} = \sum_{i=1}^n x_i \otimes e_i$ , $\hat{y} = \sum_{j=1}^n y_j \otimes e_j$

and define the sesquilinear form $\langle\cdot,\cdot\rangle : \mathcal{B}(\mathcal{H}) \bar{\otimes} \mathcal{H} \longrightarrow \mathbb{C}$ via

$$\left\langle \sum_{i=1}^n A_i \otimes x_i , \sum_{i=1}^n B_i \otimes y_i \right\rangle_K = \left\langle \hat{x}, \mathcal{M}_n\left(\hat{A}^* \hat{B}\right) \hat{y} \right\rangle_{H \otimes \mathbb{C}^n}$$

In particular we have $\left\langle \sum_{i=1}^n A_i \otimes x_i , \sum_{i=1}^n A_i \otimes x_i \right\rangle = \langle \hat{x}, \underbrace{\mathcal{M}_n(\underbrace{\hat{A}^* \hat{A}}_{\geq 0}) \hat{x}}_{\geq 0} \rangle \geq 0$

$\Rightarrow \langle\cdot,\cdot\rangle_K$ is positive semidefinit on $\mathcal{B}(\mathcal{H}) \bar{\otimes} \mathcal{H}$.

Consider $\ker(\langle\cdot,\cdot\rangle_K) := \{ \phi \in \mathcal{B}(\mathcal{H}) \bar{\otimes} \mathcal{H} : \langle \phi, \phi\rangle_K = 0 \}$

$\Rightarrow \langle\cdot,\cdot\rangle_K$ is positive definit on $\mathcal{B}(\mathcal{H}) \bar{\otimes} \mathcal{H} / \ker(\langle\cdot,\cdot\rangle_K)$

$\Rightarrow \left( \overline{\mathcal{B}(\mathcal{H}) \bar{\otimes} \mathcal{H} / \ker(\langle\cdot,\cdot\rangle_K)}^{\|\cdot\|_K} , \langle\cdot,\cdot\rangle_K \right)$ is a Hilbert space

Want representation $\pi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(K)$. $\quad K = \overline{\left(\mathcal{B}(\mathcal{H}) \bar{\otimes} \mathcal{H} \big/ \ker\langle\cdot,\cdot\rangle\right)}$

$$X \longmapsto \pi(X)$$

For any $X \in \mathcal{B}(\mathcal{H})$ define $\pi(X) : \mathcal{B}(\mathcal{H}) \bar{\otimes} \mathcal{H} \longrightarrow \mathcal{B}(\mathcal{H}) \bar{\otimes} \mathcal{H}$

via $\quad \pi(X) \sum_{i=1}^{n} A_i \otimes x_i = \sum_{i=1}^{n} (XA_i) \otimes x_i$

Show: i) if $\langle \phi, \phi \rangle = 0$, then $\langle \pi(X)\phi, \pi(X)\phi\rangle = 0$ (well defined)

ii) $\pi(X)$ is bounded wrt. $\langle\cdot,\cdot\rangle$. Then extend to $K$

By definition $\forall \phi = \sum_{i=1}^{n} A_i \otimes x_i \in \mathcal{B}(\mathcal{H}) \bar{\otimes} \mathcal{H}$:

$$\|\pi(X)\phi\|_K^2 = \langle \pi(X)\phi, \pi(X)\phi\rangle_K$$

$$= \left\langle \sum_{i=1}^{n} (XA_i) \otimes x_i , \sum_{j=1}^{n} (XA_j) \otimes x_j \right\rangle_K$$

$$= \langle \hat{x}, \mathcal{M}_n((\hat{X}\hat{A})^*(\hat{X}\hat{A}))\hat{x}\rangle_H$$

$$= \langle \hat{x}, \mathcal{M}_n(\hat{A}^*\hat{X}^*\hat{X}\hat{A})\hat{x}\rangle_H$$

$$\overset{*}{\leq} \|\hat{X}\|^2 \langle \hat{x}, \mathcal{M}_n(\hat{A}^*\hat{A})\hat{x}\rangle_H$$

$$= \|\hat{X}\|^2 \|\phi\|_K^2$$

$\Rightarrow$ ii) $\forall X \in \mathcal{B}(\mathcal{H})$: $\pi(x)$ is bounded on $\mathcal{B}(\mathcal{H}) \bar{\otimes} \mathcal{H}$

i) $\forall \phi \in \mathcal{B}(\mathcal{H}) \bar{\otimes} \mathcal{H}$: $\phi = 0 \Rightarrow \|\phi\|_K = 0 \Rightarrow \|\pi(X)\phi\|_K^2 = 0 \Rightarrow \pi(X)\phi = 0$

$\forall \phi$: $\quad \langle \phi, \hat{A}^*\hat{X}^*\hat{X}\hat{A}\phi\rangle = \langle \phi, \hat{A}^*(X^*X \otimes \mathbb{1})\hat{A}\phi\rangle$

$$= \langle \hat{A}\phi, (X^*X \otimes \mathbb{1})\hat{A}\phi\rangle$$

$X^*X \leq \|X\|^2 \mathbb{1}$ (both s.a., spectrum $\leq \|X\|^2$) $\longrightarrow$

$$\leq \langle \hat{A}\phi, (\|X\|^2 \mathbb{1} \otimes \mathbb{1})\hat{A}\phi\rangle$$

$$= \|X\|^2 \langle \phi, \hat{A}^*\hat{A}\phi\rangle$$

$\Rightarrow \mathcal{M}_n$ is positive since $\mathcal{M}$ is completely positive

Define $V : \mathcal{H} \longrightarrow K$ by $Vx = \mathbb{1} \otimes x$

- $V$ is bounded linear: 
$$\|Vx\|_K^2 = \langle \mathbb{1} \otimes x, \mathbb{1} \otimes x \rangle_K = \langle x, \mathcal{M}_n(\hat{\mathbb{1}})\, x \rangle_H \leq C \cdot \|x\|_H^2$$

Finally $\forall x \in H$, $A \in \mathcal{B}(\mathcal{H})$

$$\langle x, V^* \pi(A) V x \rangle_H = \langle Vx, \pi(A) Vx \rangle_K = \langle \mathbb{1} \otimes x, A \otimes x \rangle_K = \langle x, \mathcal{M}_n(\hat{\mathbb{1}}\, \hat{A})\, x \rangle = \langle x, \mathcal{M}(A)\, x \rangle$$

$$\Rightarrow \mathcal{M}(A) = V^* \pi(A) V$$

∎