Some Homological Algebra, the Mayer-Vietoris Sequence, Computations and Classical Applications of deRham Cohomology

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1. cochain Complexes and Cohomology

1.1. cochain Complexes

Definition 1. A **cochain complex** is a sequence of vector spaces $\{C^k\}_{k\in\mathbb{Z}}$ together with a sequence of linear maps $d_k : C^k \to C^{k+1}$ s.t. $d_{k+1} \circ d_k = 0$ (i.e. $\operatorname{im} d_k \subseteq \ker d_{k+1}$) for every $k \in \mathbb{Z}$. The subscript for d will often be dropped and we call d the **differential** or **boundary operator** of the cochain complex.

Example 1. For a smooth manifold M, the sequence of vector spaces given by $C^k = \Omega^k(M)$ and with $d_k : \Omega^k(M) \to \Omega^{k+1}(M)$ being the exterior derivative, is a cochain complex.

Definition 2. (i) A sequence of linear maps

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{1}$$

is said to be **exact at** B if $\operatorname{im} f = \ker g$.
(ii) A sequence of linear maps

$$A^0 \xrightarrow{f_0} A^1 \xrightarrow{f_1} A^2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} A^n \tag{2}$$

is said to be **exact** if it is exact at every A^k for $k \neq 0, n$.
(iii) A sequence of five linear maps of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \tag{3}$$

is called a **short exact sequence**.

Remark. (i) For a short exact sequence, as above, f is injective and g is surjective.

(ii) The sequence $0 \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if g is injective.

(iii) The sequence $A \xrightarrow{f} B \xrightarrow{g} 0$ is exact if and only if f is surjective.

1.2. Cohomology of a cochain complex

Element in ...	are generally called ...	and in deRham cohomology called ...
C^k	k-cochains	k-form
$\ker d_k$	k-cocycle	closed k-form
$\operatorname{im} d_{k-1}$	k-coboundary	exact k-form

Definition 3. Let $\mathcal{C} := (\{C^k\}_{k\in\mathbb{Z}}, \{d_k\}_{k\in\mathbb{Z}})$ be a cochain complex. Then the quotient vector space

$$H^k(\mathcal{C}) := \underbrace{\ker d_k}_{=:Z^k(\mathcal{C})} / \underbrace{\operatorname{im} d_{k-1}}_{=:B^k(\mathcal{C})} \tag{4}$$

is called the **k-th cohomology vector space of $\mathcal{C}$**. The equivalence class $[c] \in H^k(\mathcal{C})$ of a cocylce $c \in \ker d_k$ is called its **cohomology class**.

Remark. The cohomology H^k of a cochain complex is a measure for the failure of $\mathcal{C}$ to be exact at C^k.

Definition 4. Let $\mathcal{A}, \mathcal{B}$ be two cochain complexes with differentials d, d'. A **cochain map** is a sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ of linear maps $\varphi_k : A^k \to B^k$ s.t. $d'_k \circ \varphi_k = \varphi_{k+1} \circ d_k$ for every $k \in \mathbb{Z}$. As with the differential, we will drop the subscript of φ when it is clear from the context.

Remark. A cochain map φ induces a well-defined linear map $\varphi : H^k(\mathcal{A}) \to H^k(\mathcal{B})$ between the cohomology vector spaces of $\mathcal{A}$ and $\mathcal{B}$ via $\varphi[a] = [\varphi(a)]$. To see this, let $[a] \in H^k(\mathcal{A})$ i.e. let $a \in \ker d_k$. Then

$$d'_k(\varphi_k(a)) = \varphi_{k+1}(d_k(a)) = \varphi_{k+1}(0) = 0 \tag{5}$$

Hence $\varphi_k : \ker d_k \to \ker d'_k$. To show that it is well-defined, let $[a] = 0$ in $H^k(\mathcal{A})$. Then $a \in \operatorname{im} d_{k-1}$ i.e. $\exists b \in A^{k-1} : d_{k-1}(b) = a$. Hence

$$\varphi_k(d_{k-1}(b)) = d'_{k-1}(\varphi_{k-1}(b)) \in \operatorname{im} d'_{k-1}, \tag{6}$$

and thus $[\varphi(a)] = 0$ in $H^k(\mathcal{B})$.

Example 2. Let $F : N \to M$ be a smooth map. Then $F^* : \Omega^k(M) \to \Omega^k(N)$ commutes with the differential and therefore induces a map on cohomology.

1.3. Connecting Homomorphism

Definition 5. A sequence of cochain complexes

$$0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \to 0 \tag{7}$$

is called **short exact** if i and j are cochain maps and for every $k \in \mathbb{Z}$ the sequence

$$0 \to A^k \xrightarrow{i} B^k \xrightarrow{j} C^k \to 0 \tag{8}$$

is short exact.

Given the data of a short exact sequence of cochain complexes, one can construct the following linear map $\delta^* : H^k(\mathcal{C}) \to H^{k+1}(\mathcal{A})$, called the **connecting homomorphism**. Its purpose will become clear via the zig-zag-Lemma, Lemma 1.
Let $k \in \mathbb{Z}$ be arbitrary and consider the following diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A^{k+1} & \xrightarrow{i} & B^{k+1} & \xrightarrow{j} & C^{k+1} & \longrightarrow & 0 \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & A^{k} & \xrightarrow{i} & B^{k} & \xrightarrow{j} & C^{k} & \longrightarrow & 0 \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & A^{k-1} & \xrightarrow{i} & B^{k-1} & \xrightarrow{j} & C^{k-1} & \longrightarrow & 0
\end{array}
$$

Now consider the following steps:

- Let $[c]$ be an arbitrary element in $H^k(\mathcal{C})$. That is, let $c \in C^k \cap \ker d$
- Since the middle sequence in the above diagram is exact at C^k, the map j is surjective and there exists a $b \in B^k$ s.t. $j(b) = c$.
- Apply d to b to obtain $db \in B^{k+1}$.
- Since j is a chain map and $c \in \ker d$ we have $j(db) = d(j(b)) = d(c) = 0$. Hence $db \in \ker j$ and thus, since the upper sequence is exact at B^{k+1}, the element db lies in the image of i.
- Hence there exits an element $a \in A^{k+1}$ s.t. $i(a) = db$.
- In order to see that $a \in A^{k+1} \cap \ker d$, note that $i(da) = d(i(a)) = d(db) = 0$. Since the upper sequence is exact at A^{k+1}, the map i is injective and thus $da = 0$.

We define the connecting homomorphism $\delta^* : H^k(\mathcal{C}) \to H^{k+1}(\mathcal{A})$ via $\delta^*[c] = [a]$. The above is summarized in the following diagram

$$
\begin{array}{ccc}
a & \xmapsto{i} & db \\
 & & \uparrow d \\
 & & b & \xmapsto{j} & c
\end{array}
$$

or as $\delta^* = i^{-1} \circ d \circ j^{-1}$, where i^{-1} and j^{-1} is to be understood as choosing one element in the pre-image. By tracing through the argument with another $c' \in C^k \cap \ker d$ as well as $c + c'$, we note that this map is linear. In order to show that it is well defined i.e. that $[a]$ depends neither on the choice of representative

c of $[c]$, nor on the choice of pre-image b, under the map j, of that representative c (note that the choice of pre-image of db was unique since i is injective), we argue as follows:
Let $c' \in C^k \cap \ker d$ be another representative of $[c]$. Then we want show that $a - a'$ is a coboundary i.e. that there is a $z \in A^k$ s.t. $a - a' = dz$.

- Since $c - c'$ represents $[0]$ there is a $x \in C^{k-1}$ s.t. $c - c' = dx$.
- Since the sequence is exact at C^{k-1}, the map j is surjective and thus there exists a $y \in B^{k-1}$ s.t. $j(y) = x$.
- Note that the element $u := (b - b') - dy \in B^k$ lies in $\ker j$ since

$$j(dy - (b - b')) = j(dy) - j(b - b') = dx - (c - c') = 0. \tag{9}$$

- Thus, by exactness at B^k, there is a $z \in A^k$ s.t. $i(z) = u$.
- Finally,

$$i(dz-(a-a')) = d((b-b')-dy))-i(a-a')) = d(b-b')-i(a-a') = 0 \tag{10}$$

  since $a - a'$ was chosen to lie in the i-pre-image of $d(b - b')$.
- Thus, by the injectivity of i, we conclude that $dz = a - a'$.

To show well-definedness with respect to the choice of pre-image of c, let b' be another element in the pre-image of $c \in C^k$. Then $j(b - b') = c - c = 0$ and hence there exists a $u \in A^k$ such that $i(u) = b - b'$. But now

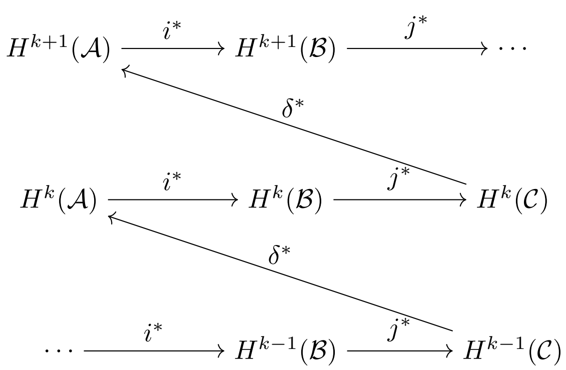
$$i(du) = d(i(u)) = d(b - b'). \tag{11}$$

Thus, since i is injective $du = c - c'$ and $c - c'$ is a coboundary and δ^* does not depend on the choice of pre-image of c.

Theorem 1. *(Zig-Zag-Lemma) Let*

$$0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \to 0 \tag{12}$$

be a short exact sequence of cochain complexes. Then the sequence



is long exact.

Proof. Let $k \in \mathbb{Z}$ be arbitrary.

- Exactness at $H^k(\mathcal{A})$ i.e. $\operatorname{im} \delta^* = \ker i^*$

  "$\subseteq$": Let $\delta^*[c] \in H^k(\mathcal{A})$. Then as in the construction of δ^* a representative of the class $\delta^*[c]$ is given by the pre-image under i of db. Thus $i^*\delta^*[c] = [i(\delta^*(c)])] = [db] = 0$.

  "$\supseteq$": Let $i^*[a] = 0$. That is, let $i(a)$ be a coboundary in B^k. Then there is a $b \in B^{k-1}$ s.t. $i(a) = db$. Applying j to this b gives $j(b) \in C^{k-1}$. Tracing back the steps in the opposite direction shows that applying δ^* to $[j(b)]$ gives $[a]$.

- Exactness at $H^k(\mathcal{B})$ i.e. $\operatorname{im} i^* = \ker j^*$

  "$\subseteq$": Let $i^*[a] \in H^k(\mathcal{B})$. Then $j^*(i^*[a]) = j^*[i(a)] = [j(i(a))] = [0]$, since (12) is exact at B^k.

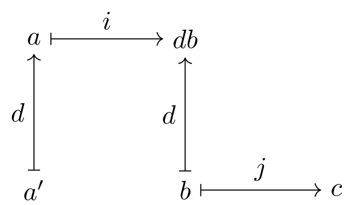
  "$\supseteq$": Let $[b] \in \ker j^*$. Then $j^*[b] = [j(b)] = 0$. Hence $j(b) = dc$ for some $c \in C^{k-1}$. Since j is surjective, there is a $b' \in B^{k-1}$ s.t. $j(b') = c$ and thus $j(b - db') = j(b) - dj(b') = 0$. Hence there is a $a \in A^k$ s.t. $i(a) = b - db'$. Thus $i^*[a] = [i(a)] = [b - db'] = [b]$, showing that $[b] \in \operatorname{im} i^*$.

- Exactness at $H^k(\mathcal{C})$ i.e. $\operatorname{im} j^* = \ker \delta^*$

  "$\subseteq$": Let $[b] \in H^k(\mathcal{B})$. Then by definition $\delta^* j^*[b] = \delta^*[j(b)]$. Now we trace the element $[j(b)]$ through the machinery for δ^*:

  We may pick b as the pre-image of $j(b)$ in B^k. Since $[b] \in H^k(\mathcal{B})$, the element b is a cocycle and thus $db = 0$. Hence $db = i(0)$ and thus by the injectivity of i we conclude $\delta^*[j(b)] = [0]$.

  "$\supseteq$": Let $\delta^*[c] = [a] = 0 \in H^k(\mathcal{A})$. Then $a = da'$ with $a' \in A^{k-1}$. On the other hand we may trace back a along the path of δ^* to an element $c \in C^{k-1}$ as

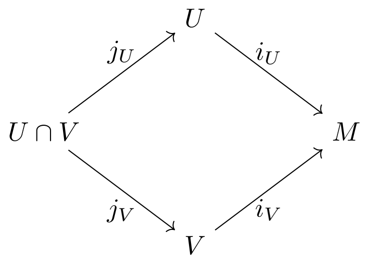
  

  But now $b - i(a')$ is a coboundary since $d(b - i(a')) = db - i(da') = 0$ and also $j(b - i(a')) = j(b) - j(i(a')) = c - 0 = c$ by the exactness at B^{k-1}.

□

1.4. Mayer-Vietoris Sequence

Let M be a manifold and let $\{U, V\}$ be an open cover of M with the following inclusion maps, forming a commutative diagram of manifolds.



For every $k \in \mathbb{Z}$, the above maps induce the sequence

$$0 \to \Omega^k(M) \xrightarrow{i} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{j} \Omega^k(U \cap V) \to 0 \tag{13}$$

defined via

$$i : \omega \mapsto (i_U^* \omega, i_V^* \omega) = (\omega|_U, \omega|_V), \quad \omega \in \Omega^k(M) \tag{14}$$

$$j : (\omega, \sigma) \mapsto j_U^* \omega - j_V^* \sigma = \omega|_{U \cap V} - \sigma|_{U \cap V}, \quad \omega \in \Omega^k(U) \oplus \Omega^k(V). \tag{15}$$

We call i the **restriction map** and j the **difference map**. Together with the respective boundary operator d, the three sequences of vector spaces (indexed by k) form cochain complexes; in the case of $\Omega^k(U) \oplus \Omega^k(V)$ this can be seen by noting that $\Omega^k(U) \oplus \Omega^k(V) \cong \Omega^k(U \coprod V)$, where $\coprod$ denotes the disjoint union, and thus $d(\omega, \sigma) = (d\omega, d\sigma)$.

Proposition 1. *The maps i and j are cochain maps.*

Proof. Let $k \in \mathbb{Z}$ be arbitrary. Let $\omega \in \Omega^k(M)$ be arbitrary. Then

$$d(i(\omega)) = (d(i_U^*(\omega), d(i_V^*(\omega)) = (i_U^*(d\omega), i_V^*(d\omega)) = i(d(\omega)). \tag{16}$$

Let $(\omega, \sigma) \in \Omega^k(U) \oplus \Omega^k(V)$ be arbitrary. Then

$$d(j(\omega, \sigma)) = d(j_U^* \omega) - d(j_V^* \sigma) = j_U^*(d\omega) - j_V^*(d\sigma) = j(d(\omega, \sigma)). \tag{17}$$

□

Proposition 2. *For every $k \in \mathbb{Z}$, the sequence*

$$0 \to \Omega^k(M) \xrightarrow{i} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{j} \Omega^k(U \cap V) \to 0 \tag{18}$$

is short exact.

Proof. Exactness at $\Omega^k(M)$ and $\Omega^k(U) \oplus \Omega^k(V)$ are clear. For the exactness at $\Omega^k(U \cap V)$ i.e. for the surjectivity of the difference map j, consider the following[1]:

[1] It is generally not true that there exists a smooth extension of $\omega \in \Omega^k(U \cap V)$ to U or V. So the naive idea of choosing such an extension η and defining $j^{-1}(\omega) = (\eta, 0)$ does not work.

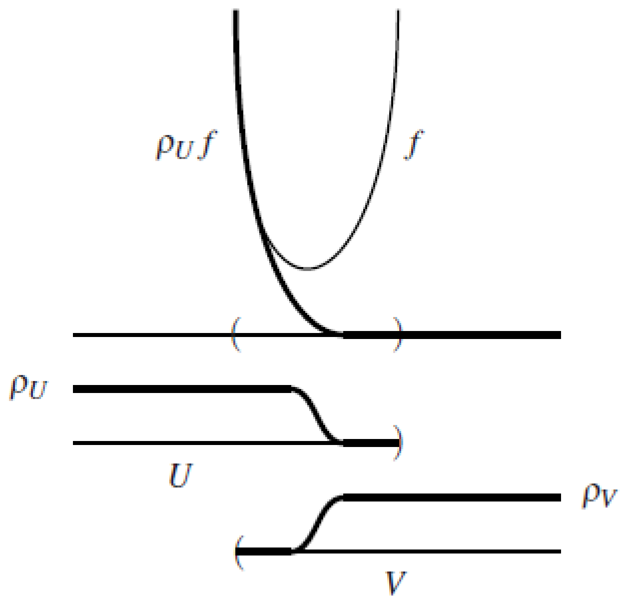


Figure 1: Rewriting a function f on $U \cap V$ as a difference of functions on U and V. Fig. 26.1 [TuMf].

Let $\omega \in \Omega^k(U \cap V)$ be arbitrary and let $\{\rho_U, \rho_V\}$ be a partition of unity subordinate to the open cover $\{U, V\}$. Then define the forms

$$\eta_U := \begin{cases} \rho_V \omega, & \text{on } U \cap V \\ 0, & \text{on } U \setminus \operatorname{supp} \rho_V \end{cases} \tag{19}$$

and

$$\eta_V := \begin{cases} \rho_U \omega, & \text{on } U \cap V \\ 0, & \text{on } V \setminus \operatorname{supp} \rho_U. \end{cases} \tag{20}$$

To see that η_U defines a smooth k-form, note that the intersection $(U \cap V) \cap (U \setminus \operatorname{supp} \rho_V) = (U \cap V) \cap (\operatorname{supp} \rho_V)^c$ is an open set, on which 0 and $\rho_V \omega$ agree. Hence they can be glues to a smooth form. The same is true for η_V.
Now we have

$$j(\eta_U, -\eta_V) = \eta_U|_{U \cap V} + \eta_V|_{U \cap V} = \rho_U \omega + \rho_V \omega = \omega, \tag{21}$$

showing that j is surjective. □

Thus, as a result the Zig-Zag-Lemma applies and we obtain a long exact sequence in cohomology, called the Mayer-Vietoris sequence:

$$\begin{array}{ccccc}
H^{k+1}(M) & \xrightarrow{i^*} & H^{k+1}(U) \oplus H^{k+1}(V) & \xrightarrow{j^*} & \cdots \\
 & \nwarrow^{\delta^*} & & & \\
H^k(M) & \xrightarrow{i^*} & H^k(U) \oplus H^k(V) & \xrightarrow{j^*} & H^k(U \cap V) \\
 & \nwarrow^{\delta^*} & & & \\
\cdots & \xrightarrow{i^*} & H^{k-1}(U) \oplus H^{k-1}(V) & \xrightarrow{j^*} & H^{k-1}(U \cap V)
\end{array}$$

Let us see explicitly what the connecting homomorphism does here:

$$\begin{array}{ccc}
\alpha \xmapsto{i} & (d\zeta_U, -d\zeta_V) & \\
 & \uparrow d & \\
 & (\zeta_U, -\zeta_V) & \xmapsto{j} \zeta
\end{array}$$

1. Let ζ be a closed $k-1$-form in $U \cap V$ and let $\{\rho_U, \rho_V\}$ be a partition of unity subordinate to $\{U, V\}$. Extend $\rho_U \zeta$ by 0 to a $k-1$-form ζ_V to all of V (same for U). Choose $(-\zeta_U, \zeta_V)$ in the pre-image of ζ under j.

2. Apply d to obtain $(-d\zeta_U, d\zeta_V)$. Since ζ is closed and j is a chain map $j(-d\zeta_U, d\zeta_V) = dj(-\zeta_U, \zeta_V) = d\zeta = 0$. Hence the difference of $-d\zeta_U$ and $d\zeta_V$ vanishes on $U \cap V$ (even though of course $\zeta_U + \zeta_V = \zeta$ which does not vanish in general.).

3. Hence $-d\zeta_U$ and $d\zeta_V$ can be glued to a global k-form, which is why $(-d\zeta_U, d\zeta_V)$ has a pre-image α under i. The form α is both an extension of $-d\zeta_U$ from U to M and of $d\zeta_V$ from V to M.

4. Since $(-d\zeta_U, d\zeta_V)$ is exact and i is a chain map and injective, α is closed.

Often the dimension of the cohomology groups alone gives a lot of information about the manifold. The following Lemma gives a restriction:

Lemma 1. *Let*

$$0 \xrightarrow{d_{-1}} A^0 \xrightarrow{d_0} A^1 \xrightarrow{\delta} \ldots \xrightarrow{d_{m-1}} A^m \xrightarrow{d_m} 0 \tag{22}$$

be a long exact sequence with $\dim A^k < \infty$ *for every* $k \in \mathbb{Z}$. *Then*

$$\sum_{k=0}^{m} (-1)^k \dim A^k = 0. \tag{23}$$

Proof. We use the rank-nullity theorem $\dim A^k = \dim \ker d_k + \dim \operatorname{im} d_k$ and the exactness of the sequence $\operatorname{im} d_k = \ker d_{k+1}$ to compute

$$
\begin{aligned}
\sum_{k=0}^{m}(-1)^k \dim A^k &= \sum_{k=0}^{m}(-1)^k(\dim\ker d_k + \dim\operatorname{im} d_k) \\
&= \sum_{k=0}^{m-1}(-1)^k(\dim\ker d_k + \dim\ker d_{k+1}) + (-1)^m \dim A^m \\
&= \dim\ker d_0 + (-1)^{m-1}\dim\ker d_m + (-1)^m \dim A^m = 0
\end{aligned}
$$

Since d_0 is injective, the first term is 0 and since $(-1)^{m-1}\dim\ker d_m + (-1)^m \dim A^m$ the second two terms vanish. □

Remark. The above Lemma 1 can be slightly weakened in that the assumption of $\dim A^k < \infty$ can be dropped for every third term in the sequence. In that case, one can conclude that those spaces are also finite dimensional. To see this, note that via rank-nullity

$$
\begin{aligned}
\dim A^k &= \dim\ker d_k + \dim\operatorname{im} d_k && (24)\\
&= \dim\operatorname{im} d_{k-1} + \dim\ker d_{k+1} && (25)\\
&\leq \dim A^{k-1} + \dim A^{k+1} < \infty. && (26)
\end{aligned}
$$

In particular, in the setting of the Mayer-Vietoris sequence, this implies that if U, V and $U \cap V$ have finite dimensional deRham cohomology, then so does M.

Proposition 3. *In the situation as above, if U, V, and $U \cap V$ are connected, then*

(i) the sequence

$$0 \to H^0(M) \xrightarrow{i} H^0(U) \oplus H^0(V) \xrightarrow{j} H^0(U \cap V) \to 0 \tag{27}$$

is short exact and M is connected.

(ii) the long exact sequence

$$0 \to H^1(M) \xrightarrow{i} H^1(U) \oplus H^1(V) \xrightarrow{j} H^1(U \cap V) \to \dots \tag{28}$$

is also exact.

Proof. (i) By the exactness of the Mayer-Vietoris sequence, we only need to show that j^* is surjective. To see this, recall that $H^0(U) \oplus H^0(V)$ and $H^0(U \cap V)$ consists of pairs of functions, each constant on U, V, and $U \cap V$, respectively, and j assigns to any pair the difference of the restrictions to $U \cap V$. Thus, any constant function on $U \cap V$ with value $a \in \mathbb{R}$ is the image of $(a, 0)$ under the map j^*.
From Lemma 1 and the exactness of (27) we get $\dim H^0(M) - 2 + 1 = 0$, thus $\dim H^0(M) = 1$ and hence that M is connected. A point-set-topological proof is of course also possible.

(ii) By (i), the map $\delta^* : H^0(U \cap V) \to H^1(M)$ is the zero map. Hence $H^0(U \cap V)$ may be replaced by the zero map. □

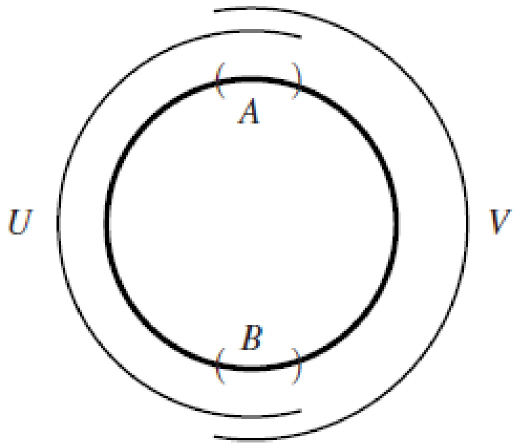


Figure 2: Covering of $\mathbb{S}^1$. Fig. 26.2 [TuMf].

2. Computations

2.1. Cohomology of $\mathbb{S}^1$

Using homotopy invariance of de Rham cohomology we note that $U \coprod V \simeq \mathbb{R} \coprod \mathbb{R}$ and $U \cap V \simeq \mathbb{R} \coprod \mathbb{R}$. Hence the second and third column are determined. For $H^0(\mathbb{S}^1) = 0$, recall that the dimension of $H^0(M)$ equals the number of connected components of M, which in this case is 1.

Thus we are left with the following table of cohomology groups:

	$\mathbb{S}^1$	$U \coprod V$	$U \cap V$
H^1	$H^1(\mathbb{S}^1)$	0	0
H^0	$\mathbb{R}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R} \oplus \mathbb{R}$

By Mayer-Vietoris we have the following exact sequence:

$$0 \to \mathbb{R} \xrightarrow{i^*} \mathbb{R} \oplus \mathbb{R} \xrightarrow{j^*} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\delta^*} H^1(\mathbb{S}^1) \to 0 \tag{29}$$

Using Lemma 1, we obtain from (29)

$$1 - 2 + 2 - \dim H^1(\mathbb{S}^1) = 0 \tag{30}$$

and thus $\dim H^1(\mathbb{S}^1) \cong \mathbb{R}$.

In order to identify a generator of $H^1(\mathbb{S}^1)$, recall from the last talk that for a closed, orientable manifold, the volume form gives a closed, but not exact form of degree $\dim(M)$. Hence in this case, $H^1(\mathbb{S}^1)$ is generated by θ.
Another way of identifying a generator of $H^1(\mathbb{S}^1)$ is to compute it explicitly: since (29) is exact at $H^1(\mathbb{S}^1)$ we have $H^1(\mathbb{S}^1) = \operatorname{im} \delta^*$.
Consider the cohomology class $[f] \in H^0(U \cap V)$, which is represented by the smooth function $f \in C^\infty(\mathbb{S}^1)$ which is 1 on the connected component containing

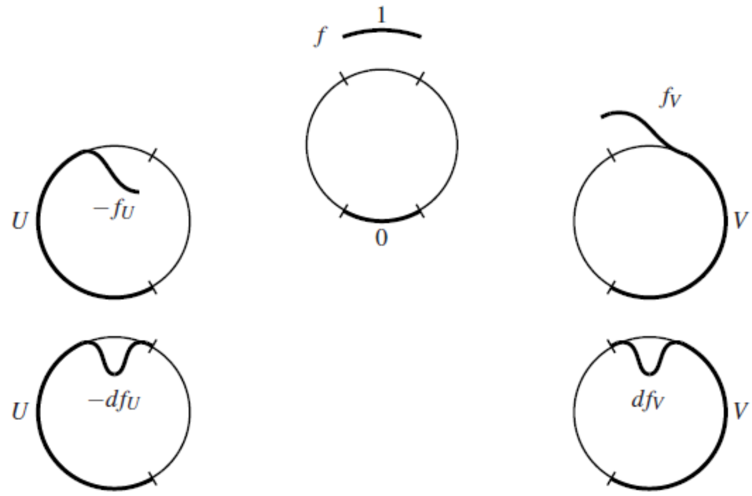


Figure 3: Connecting homomorphism $\delta^* : H^0(U \cap V) \to H^1(\mathbb{S}^1)$. Fig. 26.3 [TuMf].

the north pole and 0 on the connected component containing the south pole. Then apply $\delta^* : i^{-1} \circ d \circ j^{-1}$.

Firstly, $j^{-1}(f) = (-f_U, f_V)$ gives a function on U and V, respectively, which is an extension of $-\rho_V f$ from $U \cap V$ to U and $\rho_U f$ from $U \cap V$ to V, respectively. Applying d gives two bump functions, each supported on the connected component containing the north pole, and coinciding on all of $U \cap V$. Applying i^{-1} gives a smooth one form on $\mathbb{S}^1$, whose restriction to U and V is $-df_U$ and df_V on U and V respectively, and is thus only has support in the connected component of $U \cap V$ containing the north pole.

2.2. Cohomology of $\mathbb{S}^n$, $n \geq 2$

Theorem 2. *Let $n \geq 1$. Then*

$$H^k(\mathbb{S}^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & \textit{else.} \end{cases} \tag{31}$$

Proof. $n = 1$: see subsection 2.1
$n \Rightarrow n+1$: Assume the claim holds for $\mathbb{S}^n$. There exists an open covering of $\mathbb{S}^{n+1}$ by two discs $\mathbb{D}^n$ of dimension n (these are the two standard charts obtained by stereographic projection from the north and from the south pole). The intersection of the two is homeomorphic to $\mathbb{S}^n$. Thus, the Mayer-Vietoris sequence, the induction hypothesis, and the fact that the dimension of $H^0(\mathbb{S}^n)$ equals the number of connected components, give that the following is an exact sequence.

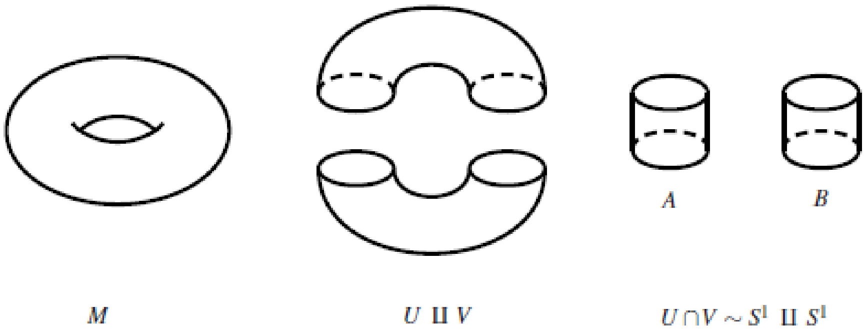


Figure 4: Covering of the torus. Fig. 28.1. [TuMf].

	$\mathbb{S}^{n+1}$	$\mathbb{D}^{n+1}\coprod\mathbb{D}^{n+1}$	$\mathbb{S}^n$
H^{n+1}	$H^{n+1}(\mathbb{S}^{n+1})$	0	0
H^{n}	$H^{n}(\mathbb{S}^{n+1})$	0	$\mathbb{R}$
H^{n-1}	$H^{n-1}(\mathbb{S}^{n+1})$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
H^{1}	$H^{1}(\mathbb{S}^{n+1})$	0	0
H^{0}	$\mathbb{R}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}$

In particular, this gives that

$$0 \to \mathbb{R} \to H^{n+1}(\mathbb{S}^{n+1}) \to 0 \tag{32}$$

and

$$0 \to 0 \to H^{k}(\mathbb{S}^{n+1}) \to 0, \quad 2 \leq k \leq n-1 \tag{33}$$

are exact and hence $H^{n+1}(\mathbb{S}^{n+1}) \cong \mathbb{R}$ and $H^k(\mathbb{S}^{n+1}) = 0$. Furthermore, Lemma 1 shows that $H^1(\mathbb{S}^{n+1}) = 0$. □

2.3. Cohomology vector space of the Torus

Choose a covering of the torus as follows:
Then A and B have the homotopy type of $\mathbb{S}^1$ and thus their cohomology is isomorphic. This gives

	$\mathbb{S}^1$	$U\coprod V$	$U\cap V$
H^2	$H^2(M)$	0	0
H^1	$H^1(\mathbb{S}^1)$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}\oplus\mathbb{R}$
H^0	$\mathbb{R}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}\oplus\mathbb{R}$

Now, Lemma 1 gives

$$1-2+2-\dim H^1(M)+2-2+\dim H^2(M) = 0 \;\Rightarrow\; \dim H^1(M) = \dim H^2(M)+1, \tag{34}$$

and furthermore the exactness of the sequence gives

$$H^2(M) = \operatorname{im}\delta_1^* \cong (\mathbb{R} \oplus \mathbb{R})/\ker\delta_1^* \cong (\mathbb{R} \oplus \mathbb{R})/\operatorname{im} j^*. \tag{35}$$

Thus the computation boils down to understanding the image of j^*. Recall that j^* is defined by

$$j^*(\omega, \eta) = (j_U^*\omega)|_{U\cap V} - (j_V^*\eta)|_{U\cap V} \tag{36}$$

where $j_U^*\omega$ and $j_V^*\eta$ are restrictions of ω and η from U, resp. V, to $U \cap V$. In our case, observe that

$$j^*(\theta_A, \theta_B) = (\theta_A - \theta_B, \theta_A - \theta_B), \tag{37}$$

which gives $\operatorname{im} j^* \cong \mathbb{R}$. Thus $H^2(M) \cong \mathbb{R}$ and hence $H^1(M) \cong \mathbb{R} \oplus \mathbb{R}$.

2.4. Cohomology ring of the Torus

In order to obtain more refined statements about the cohomology of the torus, we use the fact that one can obtain T^2 as a quotient of $\mathbb{R}^2$ (on which one knows the cohomology well). In particular, with $\Lambda := \mathbb{Z}^2$, we have

$$T^2 = \mathbb{R}^2/\Lambda. \tag{38}$$

Note that since $\pi : \mathbb{R}^2 \twoheadrightarrow T^2$ is the quotient of a smooth manifold by the smooth, proper, and free action of a discrete Lie Group Λ, the map π is a normal covering and in particular is a local diffeomorphism.

For a $\lambda \in \Lambda$ define the translation function $l_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ via $l_\lambda(q) = q + \lambda$. Then we have to following Proposition.

Proposition 4. *The pullback $\pi^* : \Omega(T^2) \to \Omega(\mathbb{R}^2)$, induced by the projection $\pi : \mathbb{R}^2 \twoheadrightarrow T^2$, is injective and*

$$\pi^*(\Omega^*(T^2)) = \{\omega \in \Omega^*(\mathbb{R}^2) : l_\lambda^*(\omega) = \omega, \forall\lambda \in \Lambda\}. \tag{39}$$

Proof. More generally, let $\pi : M \to N$ be a surjective submersion i.e. a smooth function s.t. $\pi_* : T_pM \to T_{\pi(p)}N$ is a surjection for every $p \in M$. Let $\omega \in \Omega^k(N)$ s.t. $\pi^*(\omega) = 0 \in \Omega^k(M)$, let $w_1, \ldots, w_k \in T_qN$ be arbitrary, choose a point $p \in \pi^{-1}q \subseteq M$ and define $v_i := \pi_*^{-1}(w_i)$. Then we have

$$\omega(w_1, \ldots, w_k) = \omega(\pi_*(v_1), \ldots, \pi_*(v_k)) = \pi^*(\omega)(v_1, \ldots, v_k) = 0, \tag{40}$$

i.e. $\omega = 0$.

For the "$\subseteq$" -inclusion of the characterization of the image of π^*, note that for any $\lambda \in \Lambda$ and any $q \in \mathbb{R}^2$ we have

$$(\pi \circ l_\lambda)(q) = \pi(q + \lambda) = \pi(q), \tag{41}$$

i.e. $\pi \circ l_\lambda = \pi$ and thus by functoriality $\pi^* = l_\lambda^* \circ \pi^*$, showing that for any $\omega \in \Omega(T^2)$

$$\pi^*\omega = l_\lambda^* \circ \pi^*\omega. \tag{42}$$

For the "$\supseteq$"-inclusion, assume that $\bar{\omega} \in \Omega^k(\mathbb{R}^2)$ is invariant under l_λ^* for any $\lambda \in \Lambda$. For any $p \in T^2$ and $v_1, \ldots, v_k \in T_pT^2$, define

$$\omega_p(v_1, \ldots, v_k) := \bar{\omega}_{\bar{p}}(\bar{v}_1, \ldots, \bar{v}_k) \tag{43}$$

for a choice of $\bar{p} \in \pi^{-1}(p)$ and $\bar{v}_1, \ldots, \bar{v}_k \in T_{\bar{p}}\mathbb{R}^2$ s.t. $\pi_*\bar{v}_i = v_i$. Note that if $\bar{p}$ is chosen, there is a unique choice for $\bar{v}_1, \ldots, \bar{v}_k$ since π_* is an isomorphism on each tangent space. Hence, in order to show well definedness of ω, we only need to show independence of the choice of $\bar{p}$. To do this, let $\tilde{p} = \bar{p} + \lambda$, $\lambda \in \Lambda$ be another point in $\pi^{-1}(p)$. By the invariance under l_λ^* we have

$$\bar{\omega}_{\bar{p}} = (l_\lambda^*\bar{\omega})_{\bar{p}} = l_\lambda^*(\bar{\omega}_{\bar{p}+\lambda}). \tag{44}$$

and thus

$$\bar{\omega}_{\bar{p}}(\bar{v}_1, \ldots, \bar{v}_k) = l_\lambda^*(\bar{\omega}_{\bar{p}+\lambda})(\bar{v}_1, \ldots, \bar{v}_k) \tag{45}$$

$$= \bar{\omega}_{\bar{p}+\lambda}(l_{\lambda *}\bar{v}_1, \ldots, l_{\lambda *}\bar{v}_k). \tag{46}$$

Since $\pi \circ l_\lambda = \pi$ we have $\pi_* \circ l_{\lambda *} = \pi_*$ and hence (46) shows that ω_p is independent of the choice of $\bar{p}$. Finally, by definition of ω we have

$$\bar{\omega}_{\bar{p}}(\bar{v}_1, \ldots, \bar{v}_k) = \omega_{\pi(\bar{p})}(\pi_*\bar{v}_1, \ldots, \pi_*\bar{v}_k) = (\pi^*\omega)_{\bar{p}}(\bar{v}_1, \ldots, \bar{v}_k). \tag{47}$$

Hence, $\bar{\omega} = \pi^*\omega$. □

We will now explicitly describe the ring structure on $H^*(T^2)$.

Let x, y be the standard coordinate maps on $\mathbb{R}^2$. Then for any $\lambda \in \Lambda$ we have

$$l_\lambda^*(dx) = d(l_\lambda^* x) = d(x + \lambda) = dx \tag{48}$$

and thus by Proposition 4, both dx and dy arise as pullbacks of forms α and β on T^2. Furthermore we have

$$\pi^*(d\alpha) = d(\pi^*\alpha) = d(dx) = 0. \tag{49}$$

Since π^* was injective, α is closed and thus defines a class in $H^1(T^2)$.

Proposition 5. *The forms $1, \alpha, \beta, \alpha \wedge \beta$ represent a basis of $H^*(T^2)$.*

Proof. Let $I^2 := [0,1]^2$, let $i : I^2 \hookrightarrow \mathbb{R}^2$ be the canonical inclusion and define $F := \pi \circ i : I^2 \to T^2$. Then $F^*\alpha = i^*(\pi^*\alpha) = i^*dx$ is the restriction of dx to unit square.

We compute:

$$\int_M \alpha \wedge \beta = \int_{F(I^2)} \alpha \wedge \beta = \int_{I^2} F^*(\alpha \wedge \beta) = \int_{I^2} dx \wedge dy = \int_0^1 \int_0^1 dx dy = 1. \tag{50}$$

Thus, the form $\alpha \wedge \beta$ represents a non-zero cohomology class, since otherwise by Stokes, the above integral were 0. Since $\dim H^2(T^2) = 1$ we conclude that

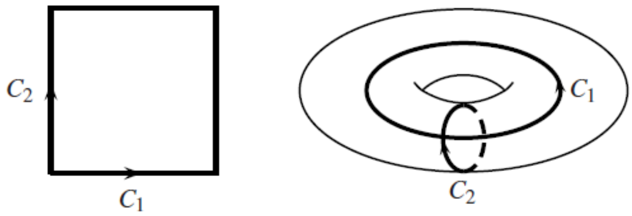


Figure 5: Parametrization and curves on the torus. Fig. 28.2. [TuMf].

$[\alpha \wedge \beta]$ forms a basis of $H^2(T^2)$.

In order to see that α, β form a basis of $H^1(T^2)$, consider the two maps $i_1, i_2 : I \to \mathbb{R}$ given by $i_1(t) = (t, 0)$ and $i_2(t) = (0, t)$. The two curves induce two closed curves C_1, C_2 in T^2 via $C_k = \pi \circ i_k$. Furthermore

$$\begin{aligned} C_1^*\alpha &= i_1^*\pi^*\alpha = i_1^* dx = d i_1^* x = dt \\ C_1^*\beta &= i_1^*\pi^*\beta = i_1^* dy = d i_1^* y = 0 \end{aligned}$$

and thus

$$\begin{aligned} \int_{C_1} \alpha &= \int_{C_1(I)} \alpha = \int_I C_1^*\alpha = \int_0^1 dt = 1 \\ \int_{C_1} \beta &= \int_{C_1(I)} \beta = \int_I C_1^*\beta = \int_0^1 0 = 0 \end{aligned}$$

A similar argument is true for C_2 giving $\int_{C_2} \beta \neq 0$. Thus, neither α nor β is exact on T^2, and hence both define non-trivial cohomology classes $[\alpha]$ and $[\beta]$. The two classes are linearly independent as otherwise $\int_{C_2} \alpha \neq 0 \Rightarrow \int_{C_2} \beta = 0$ which is not true.

Since T^2 is connected, $H^0(T^2)$ is one-dimensional and thus generated by the cohomology class induced by the constant function. □

In conclusion, the algebra $H^*(T^2)$ is isomorphic to

$$\bigwedge(a, b) = T(\mathbb{R}\,x \oplus \mathbb{R}\,y)/(x^2, y^2, xy + yx), \quad \deg(x) = \deg(y) = 1 \tag{51}$$

where $T(V)$ is the tensor algebra on the vector space V and $\mathbb{R}\,x \oplus \mathbb{R}\,y$ is the two dimensional, real vector space generated by x and y. This algebra is called the **exterior algebra of degree 1**.

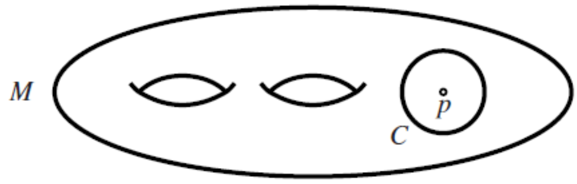


Figure 6: Punctured compact oriented surface M. Fig. 28.3. [TuMf].

2.5. Cohomology of genus g-surfaces

Lemma 2. *Let M be a compact, oriented surface, let $p \in M$ and let $i : \mathbb{S}^1 \to M \setminus \{p\}$ be the inclusion of a small circle around the puncture. Then the restriction map*

$$i^* : H^1(M \setminus \{p\}) \to H^1(\mathbb{S}^1) \tag{52}$$

is the zero map.

Proof. Let $\omega \in \Omega^1(M \setminus \{p\})$ be closed and let $D \subseteq M$ be an open disc in M bounded by $C = i(\mathbb{S}^1)$. Then

$$\int_{\mathbb{S}^1} i^*\omega = \int_{\partial(M\setminus D)} \omega = \int_{M\setminus D} \underbrace{d\omega}_{=0} = 0 \tag{53}$$

Since $H^1(\mathbb{S}^1) \cong \mathbb{R}$, with the isomorphism given by integration, this shows that $i^*[\omega] = 0$ □

Proposition 6. *Let T^2 be a torus and let $p \in T^2$. Then $A := T^2 \setminus \{p\}$ has the following cohomology:*

$$H^k(A) = \begin{cases} \mathbb{R} & , \quad k = 0 \\ \mathbb{R}^2 & , \quad k = 1 \\ 0 & , \quad k \geq 2. \end{cases} \tag{54}$$

Proof. Cover T^2 by A and an open disk U around p. Since A, U, and $U \cap V$ are connected, by Proposition 3, we may start the Mayer-Vietoris sequence with $H^1(T^2)$. Using $A \cap U \simeq \mathbb{S}^1$ and $U \simeq \{*\}$, we obtain

	T^2	$A \coprod U$	$A \cap U$
H^2	$\mathbb{R}$	$H^2(A) \oplus 0$	0
H^1	$\mathbb{R} \oplus \mathbb{R}$	$H^1(A) \oplus 0$	$\mathbb{R}$

Since $H^1(U) = 0$, the difference map j^* on the level of H^1 is simply the restriction map and by Lemma 2 this restriction is 0. Hence, on the level of H^1, the morphism i^* is an isomorphism i.e.

$$H^1(A) \cong \mathbb{R} \oplus \mathbb{R}, \tag{55}$$

and furthermore the following sequence is exact:

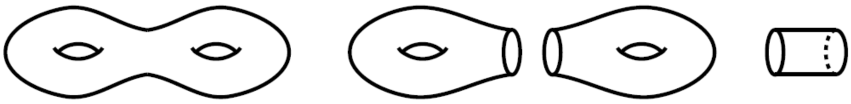


Figure 7: Covering of Σ_2. Fig. 28.4. [TuMf].

$$0 \to \mathbb{R} \to \mathbb{R} \to H^2(A) \to 0 \tag{56}$$

and thus by dimension counting (i.e. Lemma 1) we conclude $H^2(A) = 0$. □

Proposition 7. *(cohomology of genus g surface) Let $g \geq 0$ and let Σ_g denote a compact, orientable surface of genus g. Then*

$$H^k(\Sigma_g) = \begin{cases} \mathbb{R} & , \quad k = 0 \\ \mathbb{R}^{2g} & , \quad k = 1 \\ \mathbb{R} & , \quad k = 2 \\ 0 & , \quad k \geq 3. \end{cases} \tag{57}$$

Proof. The proof will proceed via induction. The base case $\Sigma_0 = \mathbb{S}^2$ is covered by subsection 2.1.

For the induction on g, assume $H^*(\Sigma_g)$ according to equation (57). Similarly to Proposition 6, let us first compute the cohomology of the punctured genus g surface $A_g := \Sigma_g \setminus \{p\}$. As for the torus,cover Σ_g by A_g and a small disc U around the puncture. Then since the A_g, U and $A_g \cap U$ are connected, by Proposition 3, we may start the Mayer-Vietoris sequence with $H^1(\Sigma_g)$. As with the torus, using that $U \simeq \{*\}$ and $A_g \cap U \simeq \mathbb{S}^1$, we get

	Σ_g	$A_g \coprod U$	$A_g \cap U$
H^2	$\mathbb{R}$	$H^2(A) \oplus 0$	0
H^1	$\mathbb{R}^{2g}$	$H^1(A) \oplus 0$	$\mathbb{R}$

.

Again, j^* is simply the restriction, which, by Lemma 2, is the 0-map. Hence we have

$$\mathbb{R}^{2g} \cong H^1(A) \tag{58}$$

and by dimension counting $H^2(A) = 0$.

Now, for the computation of $H^*(\Sigma_{g+1})$, cover Σ_{g+1} with A_g and the punctured torus A_1, with $A_g \cap A_1$ is homeomorphic to a cylinder (which is in turn homotopy equivalent to $\mathbb{S}^1$). Since Σ_{g+1} is connected, $H^0(\Sigma_{g+1}) = 0$. On the one hand, since the map $H^2(\Sigma_{g+1}) \to 0$ has as its kernel all of $H^2(\Sigma_{g+1})$, the map $\mathbb{R} \to H^2(\Sigma_{g+1})$ must be surjective. Hence $\dim(H^2(\Sigma_{g+1})) \leq 1$. On the other hand, Σ_{g+1} is an oriented closed manifold, and thus admits a volume form, which, by

Stokes, is not exact. Hence $\dim(H^2(\Sigma_{g+1}) \geq 1$ and thus $\dim(H^2(\Sigma_{g+1})) = 1$. Finally, dimension counting via Proposition 1 gives $\dim(H^1(\Sigma_{g+1})) = 2(g+1)$. □

3. Some classical applications

We want to show

Theorem 3. *(Jordan-Brouwer separation) Let $n \geq 2$ and $\Sigma \subseteq \mathbb{R}^n$ be homeomorphic to $\mathbb{S}^{n-1}$. Then*

1. *$\mathbb{R}^2 \setminus \Sigma$ has exactly 2 connected components, U_1 and U_2, one of which being bounded and one of which being unbounded,*
2. *Σ is the boundary of both U_1 and U_2.*

*We say that U_1 is the domain **inside** Σ and U_2 is the domain **outside** Σ.*

Before proving this, we need a couple of Lemmas though. Firstly, recall the Tietze extension theorem:

Theorem 4. *(Tietze Extension) Let $A \subseteq \mathbb{R}^n$ be closed and let $f : A \to \mathbb{R}^m$ be continuous, then there exists a continuous function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^m$ s.t. $\tilde{f}|_A = f$.*

Remark. The theorem is usually stated more generally with $\mathbb{R}^n$ replaced by an arbitrary normal topological space X and is actually equivalent to the normality of X.

Lemma 3. *Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be closed sets and let $\phi : A \to B$ be a homeomorphism. Then there is a homeomorphism $h : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ s.t. for every $x \in A$*

$$h(x, 0_m) = (0_n, \phi(x)). \tag{59}$$

where 0_k is the 0 in the first k components.

Proof. By the Tietz extension theorem 4 one can extend ϕ to a continuous function $\tilde{\phi} : \mathbb{R}^n \to \mathbb{R}^m$. Define firstly a homeomorphism $h_1 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ by

$$h_1(x, y) = (x, y + \tilde{\phi}(x)). \tag{60}$$

Analogously, one can extend $\psi := \phi^{-1}$ to a continuous function $\tilde{\psi} : \mathbb{R}^m \to \mathbb{R}^n$ and define $h_2 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ via

$$h_2(x, y) = (x + \tilde{\psi}(y), y). \tag{61}$$

Define $h := h_2^{-1} \circ h_1$. Then for every $x \in A$ we have

$$\begin{aligned} h(x, 0_m) &= h_2^{-1}(h_1(x, 0_m)) = h_2^{-1}(x, \tilde{\phi}(x)) && (62)\\ &= (x - \tilde{\psi}(\tilde{\phi}(x), \tilde{\phi}(x)) = (x - \psi(\phi(x)), \phi(x)) && (63)\\ &= (x - x, \phi(x)) = (0, \phi(x)). && (64) \end{aligned}$$

□

Corollary 1. *Any homeomorphism $\phi : A \to B$ between closed sets $A, B \subseteq \mathbb{R}^n$ can be extended to a homeomorphism $\tilde{\phi} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$.*

Proof. Compose the homeomorphism h from Lemma 3 with the homeomorphism which exchanges the first n components with the second n. □

Remark. Note that by restricting $\tilde{\phi}$ to $\mathbb{R}^{2n} \setminus A$ we obtain a homeomorhism $\mathbb{R}^{2n} \setminus A \to \mathbb{R}^{2n} \setminus B$. But note that this does **not** imply, and it is generally false, that $\mathbb{R}^n \setminus A \to \mathbb{R}^n \setminus B$ are homeomorphic. In fact, this would contradict the existence of the Alexander horned sphere Σ in $\mathbb{R}^3$: even though Σ is homeomorphic to $\mathbb{S}^2$, its complement, $\mathbb{R}^3 \setminus \Sigma$, is not homeomorphic to $\mathbb{R}^3 \setminus \mathbb{S}^2$, as the former is not simply connected. However, the abelianization of $\pi_1(\mathbb{R}^3 \setminus \Sigma)$ is 0, which is why the following theorem does not pose a contradiction.

Proposition 8. *Let $A \subsetneq \mathbb{R}^n$ be closed. Then we have*

$$\begin{aligned} H^{p+1}(\mathbb{R}^{n+1} \setminus A) &\cong H^p(\mathbb{R}^n \setminus A), \quad p \geq 1, \\ H^1(\mathbb{R}^{n+1} \setminus A) &\cong H^0(\mathbb{R}^n \setminus A) / \mathbb{R} \cdot 1 \\ H^0(\mathbb{R}^{n+1} \setminus A) &\cong \mathbb{R} \cdot 1. \end{aligned}$$

Proof. Identify $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ and define the following two sets

$$\begin{aligned} U_1 &:= \mathbb{R}^n \times (0, \infty) \cup (\mathbb{R}^n \setminus A) \times (-1, \infty) \\ U_2 &:= \mathbb{R}^n \times (-\infty, 0) \cup (\mathbb{R}^n \setminus A) \times (-\infty, 1) \end{aligned}$$

Then we have $U_1 \cup U_2 = \mathbb{R}^{n+1} \setminus A$ and $U_1 \cap U_2 = (\mathbb{R}^n \setminus A) \times (-1, 1)$. Define by $\phi(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n, x_{n+1} + 1)$. Then for every $x \in U_1$, the set U_1 contains a line segment from x to $\phi(x)$ and from $\phi(x)$ to a point $p \in \mathbb{R}^n \times (0, \infty)$. Maybe draw a picture with $n = 1$ to convince yourself of that. Hence U_1 is contractible (to the point p). Analogously, U_2 is contractible.

Note that $\mathbb{R}^n \setminus A$ deformation retracts to $U_1 \cap U_2$ and hence their cohomology is isomorphic. By the Mayer-Vietoris sequence we obtain an isomorphism via the connecting homomorphism

$$\delta^* : H^p(U_1 \cap U_2) \to H^{p+1}(\mathbb{R}^{n+1} \setminus A) \tag{65}$$

for $p \geq 1$. For the second isomorphism consider the following exact sequence, obtained via Mayer-Vietoris:

$$0 \longrightarrow H^0(\mathbb{R}^{n+1} \setminus A) \xrightarrow{i^*} H^0(U_1) \oplus H^0(U_2) \xrightarrow{j^*} H^0(U_1 \cap U_2) \xrightarrow{\delta^*} H^1(\mathbb{R}^{n+1} \setminus A) \longrightarrow 0$$

Elements in $H^0(U_1) \oplus H^0(U_2)$ are given by pairs of constant functions on U_1 and U_2 with values a_1 and a_2. The image of (a_1, a_2) is thus the constant function on $U_1 \cap U_2$ with value $a_1 - a_2$. Thus by the exactness ofhe Mayer-Vietoris sequence

$$\ker \delta^* = \operatorname{im} j^* = \mathbb{R} \cdot 1, \tag{66}$$

where 1 is the constant function on $U_1 \cap U_2$ with value 1. Thus we obtain

$$H^1(\mathbb{R}^{n+1} \setminus A) \cong H^0(U_1 \cap U_2)/\ker \delta^* \cong H^0(\mathbb{R}^n \setminus A)/\mathbb{R} \cdot 1. \tag{67}$$

We also have by the above Mayer-Vietoris sequence and its exactness

$$\dim H^0(\mathbb{R}^{n+1} \setminus A) = \dim(\operatorname{im} i^*) = \dim(\ker j^*) = 1 \tag{68}$$

and thus $H^0(\mathbb{R}^{n+1} \setminus A) \cong \mathbb{R}$. □

Theorem 5. *Let $A, B \subsetneq \mathbb{R}^n$ be closed subsets s.t. A and B are homeomorphic. Then*

$$H^p(\mathbb{R}^n \setminus A) \cong H^p(\mathbb{R}^n \setminus B), \quad p \geq 0. \tag{69}$$

Proof. Applying Proposition 8 $m \geq 1$ times yields

$$H^{p+m}(\mathbb{R}^{n+m} \setminus A) \cong H^p(\mathbb{R}^n \setminus A) \tag{70}$$

$$H^m(\mathbb{R}^{n+m} \setminus A) \cong H^0(\mathbb{R}^n \setminus A)/\mathbb{R} \cdot 1. \tag{71}$$

The same is true for B. By corollary 1 we know that $\mathbb{R}^{2n} \setminus A$ and $\mathbb{R}^{2n} \setminus B$ are homeomorphic and thus have the same de Rham cohomology. Thus

$$H^p(\mathbb{R}^n \setminus A) \cong H^{p+n}(\mathbb{R}^{2n} \setminus A) \cong H^{p+n}(\mathbb{R}^{2n} \setminus B) \cong H^p(\mathbb{R}^n \setminus B), \quad p \geq 1. \tag{72}$$

and

$$H^0(\mathbb{R}^n \setminus A)/\mathbb{R} \cdot 1 \cong H^n(\mathbb{R}^{2n} \setminus A) \cong H^n(\mathbb{R}^{2n} \setminus B) \cong H^0(\mathbb{R}^n \setminus B)/\mathbb{R} \cdot 1. \tag{73}$$

□

Corollary 2. *Let A, B be two closed homeomorphic subsets of $\mathbb{R}^n$. Then $\mathbb{R}^n \setminus A$ and $\mathbb{R}^n \setminus B$ have the same number of connected components.*

Proof. If $A = B = \mathbb{R}^n$ this is clear. If $A \neq \mathbb{R}^n$ and $B \neq \mathbb{R}^n$, this follows from theorem 5. If $A = \mathbb{R}^n$ but $B \neq \mathbb{R}^n$, then considering A and B as closed subsets of $\mathbb{R}^{n+1}$ and applying theorem 5 again yields

$$2 = \dim H^0(\mathbb{R}^{n+1} \setminus A) = \dim H^0(\mathbb{R}^{n+1} \setminus B) = 1 \tag{74}$$

a contradiction. Hence A and B cannot be homeomorphic to begin with. □

Now let us turn to the proof of the Jordan-Brouwer separation theorem 3:

Proof. (i) Since $\mathbb{S}^{n-1}$ is compact, so is Σ and thus Σ is closed in $\mathbb{R}^n$. Since $\mathbb{S}^{n-1}$ separates $\mathbb{R}^n$ into the two connected components

$$\text{int}(\mathbb{D}^n) = \{x \in \mathbb{R}^n : \|x\| < 1\} \quad \text{and} \quad W := \{x \in \mathbb{R}^n : \|x\| > 1\} \tag{75}$$

by corollary 2, $\mathbb{R}^n \setminus \Sigma$ also has two connected components. Furthermore, with $r := \max_{x \in \Sigma} \|x\|$, the connected set $r \cdot W$ is contained in one of the two connected components U_2 of $\mathbb{R}^n \setminus \Sigma$, which is thus unbounded. Hence for the other component, U_1, we have

$$U_1 \subseteq \mathbb{R}^n \setminus U_2 = \{x \in \mathbb{R}^n : \|x\| \leq r\}. \tag{76}$$

Thus U_1 is bounded.
(ii) Let $p \in \Sigma$ and let $V \subseteq \mathbb{R}^n$ be an arbitrary open neighborhood of p. Then the set $A := \Sigma \setminus (\Sigma \cap V)$ is closed in Σ and homeomorphically mapped to a proper, closed subset B of $\mathbb{S}^{n-1}$. Since $\mathbb{S}^{n-1}$ is closed in $\mathbb{R}^n$, the set $B = \mathbb{S}^{n-1} \cap B$ is closed in $\mathbb{R}^n$. Furthermore, since B is a proper subset of $\mathbb{S}^{n-1}$ we see that $\mathbb{R}^n \setminus B$ is connected, and thus by corollary 2 so is $\mathbb{R}^n \setminus A$. Since $\mathbb{R}^n \setminus A$ is an open subset of $\mathbb{R}^n$ and connected, it is path-connected. Hence for any $p_1 \in U_1$ and $p_2 \in U_2$ one can find a continuous curve $\gamma : [0,1] \to \mathbb{R}^n \setminus A$ s.t. $\gamma(0) = p_1$ and $\gamma(1) = p_2$. By (i), the curve γ (now considered as a curve into $\mathbb{R}^n$) has to intersect Σ, since otherwise U_1 and U_2 would lie in a common path component. The set $\gamma^{-1}(\Sigma) \subseteq [0,1]$ is closed, hence compact, and hence contains $c_1 = \min \gamma^{-1}(\Sigma)$ and $c_2 = \max \gamma^{-1}(\Sigma)$, both of which lie in $(0,1)$ since $p_1, p_2 \notin \Sigma$. Hence

$$\gamma(c_1) \in \Sigma \cap V \quad \text{and} \quad \gamma(c_2) \in \Sigma \cap V \tag{77}$$

but also

$$\gamma([0,c_1)) \subseteq U_1 \quad \text{and} \quad \gamma((c_2,1]) \subseteq U_2. \tag{78}$$

Hence there exist $t_1 \in [0,c_1)$ and $t_2 \in (c_2,1]$ s.t.

$$\gamma(t_1) \subseteq U_1 \cap V \quad \text{and} \quad \gamma(t_2) \subseteq U_2 \cap V. \tag{79}$$

showing that p is indeed a boundary point of U_1 and also of U_2. In order to see that all boundary points of U_1 have to be contained in Σ, note that since $\mathbb{R}^n \setminus \Sigma$ is an open subset of $\mathbb{R}^n$, all of its connected components are open. Hence for any $p \in U_2$ there is a neighborhood V of p, which is disjoint from U_1. The same argument holds for U_2. □

Theorem 6. *Let $A \subseteq \mathbb{R}^n$ be homeomorphic to the closed k-disk $\mathbb{D}^k$ with $k \leq n$. Then $\mathbb{R}^n \setminus A$ is connected.*

Proof. Since A is homeomorphic to $\mathbb{D}^k$, it is compact and thus closed in $\mathbb{R}^k \subset \mathbb{R}^n$. Hence by corollary 2 the number of connected components of $\mathbb{R}^n \setminus A$ coincides with that of $\mathbb{R}^n \setminus \mathbb{D}^k$, which is 1. □

Theorem 7. *(Brouwer) Let $U \subseteq \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}^n$ be continuous and injective. Then $f(U) \subseteq \mathbb{R}^n$ is open and $f : U \to f(U)$ is a homeomorphism.*

Proof. Since U is open in $\mathbb{R}^n$, it is a union of open balls $B(r,x)$ around points $x \in U$. Hence, since images preserve unions, it is sufficient to show that the images $f(B(r,x))$ are open. Let $r > 0$ and $x \in U$ be arbitrary s.t. $B(r,x) \subseteq U$ and write $D := \overline{B(r,x)}$, $S := \partial D$ and $\dot{D} := \text{int}(D) = B(r,x)$. Then since S is compact and $\mathbb{R}^n$ is Hausdorff, $\Sigma := f(S)$ is homeomorphic to S, which is homeomorphic to S^{n-1}. Thus by theorem 3, the subspace $\mathbb{R}^n \setminus \Sigma$ has two connected components, U_1 (which is bounded) and U_2 (which is unbounded); since $\mathbb{R}^n \setminus \Sigma$ is open, so are U_1 and U_2. By theorem 6, the subspace $\mathbb{R}^n \setminus f(D)$ is connected, and since it is disjoint from Σ, it must be contained in either U_1 or U_2. Since $f(D)$ is compact, the subspace $\mathbb{R}^n \setminus f(D)$ is unbounded and thus must be contained in U_2. Hence $\Sigma \cup U_1 = \mathbb{R}^n \setminus U_1 \subseteq f(D)$. Hence $U_1 \subseteq f(\dot{D})$. Since $\dot{D}$ is connected and thus $f(\dot{D})$ is also connected, and furthermore $f(\dot{D}) \subseteq U_1 \cup U_2$ we conclude that $f(\dot{D}) \subseteq U_1$ since otherwise $U_1 \subseteq U_2$. Thus $U_1 = f(\dot{D})$, which is open.

Let $W \subseteq U$ be an open subset. Then by restricting f to W and applying the same argument as above we see that $f(W)$ is also open. Hence f is a continuous, open bijection i.e. a homeomorphism. □

Corollary 3. *(Invariance of Domain) Let $A \subseteq \mathbb{R}^n$ have the subspace topology induced by $\mathbb{R}^n$ and be homeomorphic to an open subset U of $\mathbb{R}^n$. Then A is open in $\mathbb{R}^n$.*

Proof. Follows by applying Theorem 7 to U. □

Corollary 4. *(Invariance of Dimension) Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be non-empty open subsets. If U and V are homeomorphic, then $n = m$.*

Proof. Assume that $m < n$ and consider V as a (not necessarily open) subset of $\mathbb{R}^n$ via $V \subseteq \mathbb{R}^m \subseteq \mathbb{R}^n$ and topology induced by $\mathbb{R}^n$ (or equivalently $\mathbb{R}^m$). Since V is homeomorphic to U by assumption, corollary 3 implies that V is open an open subset of $\mathbb{R}^n$. This is a contradiction since V is contained in a proper linear subspace of $\mathbb{R}^n$. □

References

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