Workshop - Advances in Analysis

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The following is based on (Leoni, 2009), chapter 1.

Definition 1. Let $E \subseteq \mathbb{R}$ be a subset of the real numbers. A function $u : E \to \mathbb{R}$ is called

(i) **increasing** if $\forall x, y \in E : x < y \Rightarrow u(x) \leq u(y)$

(ii) **strictly increasing** if $\forall x, y \in E : x < y \Rightarrow u(x) < u(y)$

(decreasing analogously). In general, a function u as above is simply called **monotone** if any of the above conditions holds.

Of course, a monotone function is not continuous in general. In fact, it usually is not e.g.

$$E = [0, 1], \quad u(x) = \begin{cases} x & x \in [0, 1/2] \\ x + 1 & x \in (1/2, 1] \end{cases} \tag{1}$$

However, on intervals, it still has very nice properties regarding continuity. The definition of an interval used here is the following:

Definition 2. A subset $I \subseteq \mathbb{R}$ is called an **interval**, if $\forall x, y \in I : x < y \Rightarrow [x, y] \subseteq I$.

The following two theorems make precise what we mean by "very nice properties regarding continuity".

Theorem 1. *Let $I \subseteq \mathbb{R}$ be an interval and let $u : I \to \mathbb{R}$ be a monotone function. Then the set of points $x \in \mathbb{R}$, where u is not continuous is countable.*

Theorem 2. *Conversely, let $E \subseteq \mathbb{R}$ be a countable subset. Then there exists a monotone function $u : \mathbb{R} \to \mathbb{R}$ s.t. the set of points $x \in \mathbb{R}$ where u is discontinuous is precisely E.*

We will firstly give a preliminary definition and result:

Definition 3. Let X be a set and let $u : X \to [0, \infty]$. Then the **infinite sum of v over X** is defined as

$$\sum_{x \in X} u(x) := \sup \left\{ \sum_{x \in E} u(x) | E \subseteq X, E \text{ is finite} \right\} \tag{2}$$

The definition may be interpreted as a generalization of a real valued series. In fact, if such an infinite sum is finite, it can be written as a series i.e. then the subset $\{x \in X | u(x) > 0\}$ is countable, making the sum countable:

Lemma 1. *Let X be a set and let $u : X \to [0, \infty]$. If*

$$\sum_{x \in X} u(x) < \infty \tag{3}$$

then $\{x \in X | u(x) > 0\}$ is countable.

Proof. Let L be an infinite sum that is finite:

$$L := \sum_{x \in X} u(x) < \infty \quad . \tag{4}$$

Also let $n \in \mathbb{N}_{>0}$ and define $X_n := \{x \in X | u(x) > 1/n\}$ and let E be a finite subset of X_n. Then for a fixed $n \in \mathbb{N}_{>0}$ and fixed $E \subseteq X_n$

$$|E| \cdot \frac{1}{n} \leq \sum_{x \in E} u(x) \leq L \tag{5}$$

Hence, the size of E is bound above by nL (or more precisely by the largest integer smaller or equal to nL). Thus for any $n \in \mathbb{N}_{>0}$ the set X_n is finite as well[1]. Now note that $(X_n)_{n \in \mathbb{N}_{>0}}$ defines a monotone increasing sequence of sets $X_1 \subseteq X_2 \subseteq \ldots$ with the (set theoretic[2]) limit being $\{x \in X | u(x) > 0\}$ as n tends towards $+\infty$. As a limit of finite sets,

$$\{x \in X | u(x) > 0\} = \bigcup_{n=1}^{\infty} X_n \tag{6}$$

is thus countable. □

Thus if $L < \infty$ implies

$$L = \sum_{x \in X} u(x) = \sum_{x \in \{x \in X | u(x) > 0\}} u(x) \tag{7}$$

is a series in the usual sense.
Now for the proof of theorem 1.

For a closed interval

Proof. Let $I = [a, b] \subseteq \mathbb{R}$ with $a, b \in \mathbb{R}$ be a closed interval and let $u : I \to \mathbb{R}$ be a monotone function. Without loss of generality we assume u to be increasing. Let

$$u_-(x) := \lim_{y \to x, y \leq x} u(y) \quad \text{and} \quad u_+(x) := \lim_{y \to x, y \geq x} u(y) \tag{8}$$

i.e. the limit of u approaching x from the left and right, respectively.
Then for any $x \in (a, b)$ the **jump** of u at the point $x \in I$ is defined as

$$S_u(x) := u_+(x) - u_-(x) \tag{9}$$

[1] Were X_n not finite, then for any $E \subseteq X_n$ there exists an $E' = E \cup \{x\}$ where $x \in X_n \setminus E$.
[2] For (increasing) monotone sequences this is defined (analogously to series) as $\lim_{m \to +\infty} \bigcup_{n=1}^{m} X_n$. In the monotone decreasing case it is $\lim_{m \to +\infty} \cap_{n=1}^{m} X_n$.

Note that u is continuous as x if and only if $S_u(x) = 0$. Since u is monotone increasing $\forall x \in I : S_u(x) \geq 0$ i.e. $u_+(x) \geq u_-(x)$. Now, consider a finite subset $E := \{x_1, \ldots, x_n\} \subseteq I$, where $a \leq x_i < x_{i+1} \leq b$. Then

$$u(a) \leq u_-(x_1) \leq u_+(x_1) \leq u_-(x_2) \leq \ldots \leq u_-(x_n) \leq u_+(x_n) \leq u(b) \tag{10}$$

Hence for any finite $E \subseteq I$

$$\sum_{x \in E} S_u(x) \leq u(b) - u(a) \tag{11}$$

Hence for any E there is an upper bound $u(b) - u(a)$ to the infinite sum of u over E. With the definitions above

$$\sum_{x \in I} S(x) = \sup_E \sum_{x \in E} S(x) \leq \sup_E u(b) - u(a) \leq u(b) - u(a) < \infty \tag{12}$$

and thus by lemma 1 the set of discontinuities $\{x \in I | S_u(x) > 0\}$ is countable. □

For an arbitrary interval

Lemma 2. *Let $I \subseteq \mathbb{R}$ be an arbitrary interval. Then there exists a sequence of closed intervals $I_n = [a_n, b_n] \subseteq I \subseteq \mathbb{R}$ with $n \in \mathbb{N}$ such that*

$$\bigcup_{n=1}^{\infty} I_n = I \tag{13}$$

Note that this is a set theoretic limit of sets.

Proof. (1) Let $I = [a, b] \subseteq \mathbb{R}$ be a closed interval. Then define $I_n := [a, b]$.

(2) Let $I = (a, b) \subseteq \mathbb{R}$ be an open interval. Then define $I_n := [a+1/2^n, b-1/2^n]$ [3]. By construction, for every $n \in \mathbb{N}$ the interval I_n is closed and by the definition of an interval made earlier I_n is contained in I and the union over all the I_n (the set containing $x \in \mathbb{R}$ such that x is an element of at least one of the I_n) is precisely $\{x \in \mathbb{R} | a < x < b\}$.

(3) Let $I = [a, b) \subseteq \mathbb{R}$ be a half open interval. Then define $I_n := [a, b - 1/2^n]$ [4] with a similar argument as above.

□

With this lemma we can finalize the previous result to arbitrary intervals as follows:

Proof. Let $u : I \to \mathbb{R}$ be defined on an arbitrary interval. Then choose a sequence of intervals $I_n = [a_n, b_n] \subseteq I \subseteq \mathbb{R}$ with $\bigcup_{n=1}^{\infty} I_n = I$ where u has only countable many points of discontinuity. Denote by E_n the set of discontinuous points in I_n. Then $\bigcup_{n=1}^{\infty} E_n$ is a countable union of countable sets, and thus at most countable. □

[3] Strictly speaking, for this expression to be well defined, one needs to require $n \geq n_0$ where $1/2^n \leq \frac{b-a}{2}$, since then indeed $a + 1/2^n \leq b - 1/2^n$

[4] Again, for well definition one requires $1/2^n \leq b - a$

Proof of Converse Theorem

Proof. Now, let $E \subseteq I$ be a countable subset of an interval. If E is finite, a monotone function with discontinuities at E can be constructed by hand. Thus, consider the case where E is countably infinite: $E = \{x_1, x_2, \ldots\}$. For each $n \in \mathbb{N}$ define the (non-stricly) increasing function $u_n : \mathbb{R} \to \mathbb{R}$ as

$$u_n(x) := \begin{cases} -1/2^n & x < x_n \\ +1/2^n & x \geq x_n \end{cases} \tag{14}$$

Note that u_n is constant at every point except x_n, where it is discontinuous. Define

$$u(x) := \sum_{n=1}^{\infty} u_n(x), \quad x \in \mathbb{R} \quad . \tag{15}$$

Indeed, $u(x)$ is continuous wherever all the u_n are. To see this, consider

$$\lim_{n\to\infty} \parallel u - \sum_{k=1}^{n} u_n \parallel_\infty = \lim_{n\to\infty} \sup_{x\in I} | \sum_{k=n+1}^{\infty} u_n(x)| \leq \tag{16}$$

$$\lim_{n\to\infty} \sup_{x\in I} \sum_{k=n+1}^{\infty} |u_n(x)| = \tag{17}$$

$$\lim_{n\to\infty} \sum_{k=n+1}^{\infty} |u_n(x)| \tag{18}$$

Now since $\forall x \in \mathbb{R} : |u(x) - u_n(x)| \leq 1/2^n$, this series is dominated by $\sum_{k=n+1}^{\infty} \frac{1}{2^n}$, the limit of which is 0. Thus the sum of the u_n converges uniformly towards u and thus u is continuous at every point where all the u_n are continuous. In particular u is continuous in $\mathbb{R} \setminus E$.
To show that E constitutes all the points at which u is discontinuous write for every $m \in \mathbb{N}$

$$u(x) = u_m(x) + \sum_{n\neq m} u_n(x) \quad . \tag{19}$$

Then $\sum_{n\neq m} u_n(x)$ is continuous at u_m whereas x_m is not. Hence

$$S_u(x_m) = \underbrace{S_{u_m}(x_m)}_{>0} + \underbrace{S_{\sum_{n\neq m} u_n}(x_m)}_{=0} \tag{20}$$

i.e. $S_u(x)$ is not continuous at x_m. Thus u is discontinuous precisely at $E = \{x_1, x_2, \ldots\}$. □

Corollary 1. *There exists a monotone function $u : \mathbb{R} \to \mathbb{R}$ s.t. u is continuous on the set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous on the set of rational numbers $\mathbb{Q}$.*

Theorem 3 (Inverse of an increasing function)**.** *Let $I \subset \mathbb{R}$ be an interval bounded from below, let $u : I \to \mathbb{R}$ be an increasing function, let $J \subseteq \mathbb{R}$ be the smallest interval containing $u(I)$, and let $v : J \to \mathbb{R}$ be defined by*

$$v(y) := \inf\{z \in I | u(z) \geq y\}, \quad y \in J \tag{21}$$

Then

(i) $v : J \to \mathbb{R}$ *is increasing and left continuous,*

(ii) v *has a jump at a point* $y_0 \in J \backslash \{\sup_{x \in I} u(x)\}$ *if and only if* $\forall x \in (x_1, x_2) \subseteq I : u(x) = y_0$ *with* $x_1 < x_2$*,*

(iii) $\forall x \in I : v(u(x)) \leq x$*. Strict inequality holds if and only if* u *is constant on some interval* $[z, x] \subset I$ *with* $z < x$*,*

(iv) $v(y) = x_0$ *for all* y *in some open interval* $(y_1, y_2) \subset J$ *with* $y_1 < y_2$*, and for some* $x_0 \in I^o$ *if and only if* u *jumps at* x_0 *and* $(y_1, y_2) \subseteq (u_-(x_0), u_+(x_0))$*.*

In particular, if the function u *is stricly increasing, then* v *is a left inverse of* u *and* v *is continuous.*

References

Leoni, G. (2009). *A first course in sobolev spaces* (Vol. 105). American Mathematical Society.