Commutativity of Limits and Colimits

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Recall: Product category Let $\mathcal{C}, \mathcal{D}$ be categories. Then their product is defined by the following data

- $\mathbf{Ob}(\mathcal{C} \times \mathcal{D})$: Collection of ordered pairs (A, B) where $A \in \mathbf{Ob}(\mathcal{C})$, $B \in \mathbf{Ob}(\mathcal{D})$.
- $\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((A_1, B_1), (A_2, B_2))$: Collection of pairs of morphisms (f, g) where $f : A_1 \to A_2$, $g : B_1 \to B_2$.

The composition is defined component-wise, and the identity is the component-wise identity.

Recall: Limit Let I be a small category and $X : I \to \mathcal{C}$ a diagram, then a limit is a representation of $\mathrm{Nat}(\Delta(-), X) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ i.e. a cone $(L, (l_i)_I)$ where $L \in \mathcal{C}$ and the l_i are the components of the natural transformation $\Delta(L) \to X$, which is final among such cones.

Limit as a functor

Let $X : I \times J \to \mathcal{C}$ be a functor, then $X(i, -) : J \to \mathcal{C}$ via $j \mapsto X(i, j)$ and $f \mapsto X(id_i, f)$ is a functor[1] for every $i \in I$. (cf. a bilinear map is linear in either entry when the other is fixed.). If its limit exists, denote it L_i or $\lim_J X(i, -)$ and its legs by l^i_j or similar suggestive notation.

Theorem 1 (Herrlich und Strecker [1979])**.** *In the above context, there exists a unique functor $L : I \to \mathcal{C}$ via $i \mapsto L_i$ and $m \mapsto L_m$ s.t. for every $m : i \to \hat{\imath}$ the following diagram commutes.*

$$\begin{tikzcd}
L_i \arrow[r, "l^i_j"] \arrow[d, "L_m"] & X(i,j) \arrow[d, "X(m,j)"] \\
L_{\hat{\imath}} \arrow[r, "l^{\hat{\imath}}_j"] & X(\hat{\imath},j)
\end{tikzcd}$$

Proof. Since L_i is a limit of $X(i, -)$ and $L_{\hat{\imath}}$ is a limit of $X(\hat{\imath}, -)$ and $m : i \to \hat{\imath}$, $q : j \to \hat{\jmath}$, the assumed data gives the diagram

$$\begin{tikzcd}
L_i \arrow[r, "l^i_j"] & X(i,j) \arrow[r, "X(i,q)"] \arrow[d, "X(m,j)"] & X(i,\hat{\jmath}) \arrow[d, "X(m,\hat{\jmath})"] \\
L_{\hat{\imath}} \arrow[r, "l^{\hat{\imath}}_j"] & X(\hat{\imath},j) \arrow[r, "X(\hat{\imath},q)"] & X(\hat{\imath},\hat{\jmath})
\end{tikzcd}$$

[1] $X(id_i, f)$ is sometimes simply denoted $X(i, f)$.

Since the right square in diagram commutes, this makes $(L_i, (X(m,j) \circ l^i_j))$ a cone over $X(\hat{\imath}, -)$ and thus gives a unique morphism $L_m : L_i \to L_{\hat{\imath}}$ which makes the diagram commute.

This assignment is indeed a functor:

$$\begin{tikzcd}
L_i \arrow[r, "L_m"] \arrow[rr, bend left, "L_{n\circ m}"] \arrow[d, "l^i_j"] & L_{\hat{\imath}} \arrow[r, "L_n"] \arrow[d, "l^{\hat{\imath}}_j"] & L_{\hat{\imath}'} \arrow[d, "l^{i'}_j"] \\
X(i,j) \arrow[r, "X(m,j)"] \arrow[rr, bend right, "X(n\circ m,j)"'] & X(\hat{\imath},j) \arrow[r, "X(n,j)"] & X(\hat{\imath}',j)
\end{tikzcd}$$

Figure 1: Composition

$$\begin{tikzcd}
L_i \arrow[r, "l^i_j"] \arrow[d, "L_{\mathrm{id}_i}"] & X(i,j) \arrow[d, "X(\mathrm{id}_i,j)"] \\
L_i \arrow[r, "l^i_j"] & X(i,j)
\end{tikzcd}$$

Figure 2: Identity

In diagram 1, with $n : \hat{\imath} \to \hat{\imath}'$, both squares are commutative and thus the whole square is commutative. Now since

$$l^{i'}_j \circ L_n \circ L_m = \underbrace{X(n,j) \circ X(m,j)}_{=X(n\circ m,j)} \circ l^i_j = X(n \circ m, j) \circ l^i_j = l^{i'}_j \circ L_{n\circ m} \tag{1}$$

Now since $L_{\hat{\imath}'}$ is a limit and $X(n \circ m, j) \circ l^i_j$ makes L_i into a limit over $X(\hat{\imath}', -)$ there exists a unique morphism $L_i \to L_{\hat{\imath}'}$ making the diagram commute. Thus indeed $L_n \circ L_m = L_{n\circ m}$.

In the right square we have

$$X(\mathrm{id}_i, j) \circ l^i_j = l^i_j = l^i_j \circ \mathrm{id}_{L_i} = l^i_j \circ L_{\mathrm{id}_i} \tag{2}$$

which concludes the proof of functoriality of L. □

An analogous proof can be made for colimits instead of limits.

1 Commutation of Limits-Limit

Theorem 2 (Herrlich und Strecker [1979]). *Let $X : I \times J \to \mathcal{C}$ a diagram in $\mathcal{C}$ where I and J are small. Assume $X(i,-) : J \to \mathcal{C}$ has a limit $(L_i, (l^i_j)_J)$ for every $i \in I$ and let $L : I \to \mathcal{C}$ by $i \mapsto L_i$ be the aforementioned functor. Then the diagram X has a limit if and only if the diagram L has a limit. If they exist, they coincide.*

In other notation that means that $\lim_I \lim_J X \simeq \lim_{I\times J} X \simeq \lim_J \lim_I X$.

Proof. Let $(\overline{L}, (p_i)_I)$ be a limit of L. Then $(L, (l^i_j \circ p_i)_{I\times J})$ is a cone for the diagram X. Consider the diagram

$$
\begin{tikzcd}
 & & X(i,j) \arrow[d, "X(i,n)"] \\
 & L_i \arrow[ur, "l^i_j"] \arrow[r, "l^i_{\hat{j}}"] \arrow[d, "L_m"] & X(i,\hat{j}) \arrow[d, "X(m,\hat{j})"] \\
\overline{L} \arrow[ur, "p_i"] \arrow[r, "p_{\hat{i}}"'] & L_{\hat{i}} \arrow[r, "l^{\hat{i}}_{\hat{j}}"'] & X(\hat{i},\hat{j})
\end{tikzcd}
$$

where $m : i \to \hat{i}$, $n : j \to \hat{j}$.
Here, the lower left triangle commutes because $(\overline{L}, (p_i)_I)$ is a limit of L, the upper right triangle commutes because $(L_i, (l^i_j)_J)$ is a limit of $X(i,-)$ and the lower right square commutes by theorem 1.

Let $(R, (q_{ij})_{I\times J})$ be a cone over X, then it is in particular a cone over $X(i,-)$ for every $i \in I$. So since $(L_i, (l^i_j)_J)$ is a limit for $X(i,-)$ there exists a unique morphism $r_i : R \to L_i$ s.t. the following diagram commutes:

$$
\begin{tikzcd}
 & & X(i,j) \arrow[dd, "X(i,n)"] \\
R \arrow[r, dashed, "\exists! r_i"] \arrow[urr, bend left, "q_{ij}"] \arrow[drr, bend right, "q_{i\hat{j}}"'] & L_i \arrow[ur, "l^i_j"] \arrow[dr, "l^i_{\hat{j}}"'] & \\
 & & X(i,\hat{j})
\end{tikzcd}
$$

Thus we are in the situation to use theorem 1 for the next diagram.

$$
\begin{tikzcd}
 & & & X(i,j) \arrow[dddd, "X(m,j)"] \\
 & & L_i \arrow[ur, "l^i_j"] \arrow[dd, "L_m"] & \\
R \arrow[r, dashed, "\exists! \overline{r}"] \arrow[uurrr, bend left, "q_{ij}"] \arrow[urr, bend left, "r_i"] \arrow[drr, bend right, "r_{\hat{i}}"'] \arrow[ddrrr, bend right, "q_{\hat{i}j}"'] & \overline{L} \arrow[ur, "p_i"] \arrow[dr, "p_{\hat{i}}"'] & & \\
 & & L_{\hat{i}} \arrow[dr, "l^{\hat{i}}_j"'] & \\
 & & & X(\hat{i},j)
\end{tikzcd}
$$

By the previous arguments all the morphisms given exist and the indicated cells commute. Now it is just to show that the triangle $(R, L_i, L_{\hat{i}})$ commutes. If this is the case, we can conclude that $(R, (r_i)_I)$ is a cone over L and thus there uniquely exists an $h : R \to \overline{L}$ s.t. the diagram commutes. For this,

$$l^{\hat{i}}_j \circ L_m \circ r_i = X(m,j) \circ l^i_j \circ r_i = X(m,j) \circ q_{ij} = q_{\hat{i}j} = l^{\hat{i}}_j \circ r_{\hat{i}} \tag{3}$$

Hence since $L_{\hat{i}}$ is a limit over $X(\hat{i}, -)$ and R is a cone over that diagram, the morphism $R \to L_{\hat{i}}$ making the diagram commute is unique. Hence indeed $L_m \circ r_i = r_{\hat{i}}$. This shows that the triangle $(R, L_i, L_{\hat{i}})$ commutes.

Thus we have shown that for any cone $(R, (q_{ij})_{I\times J})$ over X there uniquely exists an $\overline{r} : R \to \overline{L}$ with the property that for every $(i,j) \in I \times J$ we have $q_{ij} = l^i_j \circ p_i \circ \overline{r}$. Consequently, $(\overline{L}, (l^i_j \circ p_i)_{I\times J})$ is a limit of X.

$\square$

An analogous proof can be done for colimits and thus also: colimits commute with colimits.

Example 1 (Pushout-Cokernel, Brandenburg [2016] 6.6.14)**.** In the category **Ab** of abelian groups and group homomorphisms, for $U_1 \subseteq A_1, U_2 \subseteq A_2$ we have the following (canonical) isomorphism

$$(A_1 \oplus A_2)/(U_1 \oplus U_2) \cong A_1/U_1 \oplus A_2/U_2 \quad . \tag{4}$$

The existence of this isomorphism is ensured by the preceding theorem. Explicitly, the construction is as follows:
Consider the finite categories

$$I : \bullet_1 \qquad \bullet_2, \qquad\qquad J : \bullet_2 \leftarrow \bullet_0 \rightarrow \bullet_1 \tag{5}$$

i.e. the index categories for a coproduct and pushout, respectively. Now consider the product category and the diagram

$$\begin{tikzcd}
U_1 \arrow[rr, "f_1"] \arrow[dd, "\exists!"'] & & A_1 & \\
& U_2 \arrow[rr, "f_2"] \arrow[dd, "\exists!"'] & & A_2 \\
\{e\} & & & \\
& \{e\} & &
\end{tikzcd}$$

Here, the theorem comes into play:

1. Either firstly compute the cokernel and then the direct sum of the two. This amounts to computing $\operatorname{colim}_J X(1,-) \cong A_1/U_1$ and $\operatorname{colim}_J X(2,-) \cong A_2/U_2$ and then their direct sum, which is $\operatorname{colim}_I \operatorname{colim}_J X = A_1/U_1 \oplus A_2/U_2$.
2. Or firstly compute the direct sums and then their cokernel. This amounts to firstly computing $\operatorname{colim}_I X(-,U) \cong U_1 \oplus U_2$ and $\operatorname{colim}_I X(-,A) \cong A_1 \oplus A_2$ and then their cokernel $\operatorname{colim}_J \operatorname{colim}_I X \cong (A_1 \oplus A_2)/(U_1 \oplus U_2)$.

The respective limit cones are given in the following diagrams.

$$
\begin{tikzcd}
U_1 \arrow[rr, "f_1"] \arrow[dd, "\exists!"'] & & A_1 \arrow[dd] & & \\
& U_2 \arrow[rrr, "f_2"] \arrow[ddd, "\exists!"'] & & & A_2 \arrow[ddd] \\
\{e\} \arrow[rr] & & A_1/U_1 \arrow[dr] & & \\
& & & A_1/U_1 \oplus A_2/U_2 & \\
& \{e\} \arrow[rrr] & & & A_2/U_2 \arrow[ul]
\end{tikzcd}
$$

Computation 1

$$
\begin{tikzcd}
U_1 \arrow[rr, "f_1"] \arrow[dr] \arrow[ddd, "\exists!"'] & & A_1 \arrow[dr] & & \\
& U_1 \oplus U_2 \arrow[rr, "f_1 \oplus f_2"] \arrow[ddd, "\exists!"'] & & A_1 \oplus A_2 \arrow[ddd] & \\
& & U_2 \arrow[ul] \arrow[rr, "f_2"] \arrow[dddd, "\exists!"'] & & A_2 \arrow[ul] \\
\{e\} \arrow[dr] & & & & \\
& \{e\} \arrow[rr] & & (A_1 \oplus A_2)/(U_1 \oplus U_2) & \\
& & \{e\} \arrow[ul] & &
\end{tikzcd}
$$

Computation 2

2 Commutation of Limit-Colimit

Theorem 3. *Let $X : I \times J \to \mathcal{C}$ a diagram s.t.* $\operatorname{colim}_I \lim_J X$ *and* $\lim_J \operatorname{colim}_I X$ *exist. Then there exists a canonical map*

$$\kappa : \operatorname*{colim}_I \lim_J X \to \lim_J \operatorname*{colim}_I X \tag{6}$$

Proof. Firstly, by the definition of a colimit, for every $i \in I$ there exists a morphism $\iota_i : \lim_J X(i,-) \to \operatorname{colim}_I \lim_J X$. Furthermore, for every fixed $i \in I$ and $j \in J$ there exists a $l^i_j : \lim_J X(i,-) \to X(i,j)$. The analogous is true for the right side of the below diagram.

Then define $\kappa_{ij} : \lim_J X(i,-) \to \operatorname{colim}_I X(-,j)$. If we can show that (1) $\lim_J X(i,-)$ is a cone over the diagram $\operatorname{colim}_I X(-,-) : J \to \mathcal{C}$ by the definition of limit, for every $i \in I$, the set $\{\kappa_{ij}\}$ uniquely induces a $\kappa_i : \lim_J X(i,-) \to \lim_J \operatorname*{colim}_I X(i,j)$ s.t. the corresponding triangle commutes. Then (2), in turn, the set $\{\kappa_i\}$ uniquely induces a morphism $\kappa : \operatorname*{colim}_I \lim_J X(i,j) \to \lim_J \operatorname*{colim}_I X(i,j)$ by the definition of a colimit.

(1) Note that it suffices to show that the right lower triangle commutes. In order to use (the here stated version of) theorem 1, we will show that lower left triangle commutes.

    It is to show that for every $\hat{i} \in I$ and $m : i \to \hat{i}$ there is a morphism $a : \lim_J X(\hat{i},-) \to X(i,j)$ s.t. the triangle commutes (for the right triangle this is b). Such an a is given by $l^{\hat{i}}_j \circ X(m,j) : \lim_J X(\hat{i},-) \to X(\hat{i},j) \to X(i,j)$ by theorem 1. The analogous result is that the right triangle commutes and thus $\lim_J X(i,-)$ is indeed a desired cone.

(2) The triangle that should commute is given by three commutative triangles.

$$
\begin{tikzcd}
\operatorname{colim}_I \lim_J X \arrow[rrrr, bend left, "\kappa"] \arrow[rrrr, phantom, "\circlearrowleft"] & & & & \lim_J \operatorname{colim}_I X \arrow[dl, "p_i"] \\
& \lim_J X(i,-) \arrow[ul, "\iota_i"] \arrow[urrr, bend left, "\kappa_i"] \arrow[rr, bend left, "\kappa_{ij}"] \arrow[rr, phantom, "\circlearrowleft"] \arrow[dr, "l^i_j"] \arrow[dd, "\lim_J X(m,-)"'] & & \operatorname{colim}_I X(-,j) & \\
& \arrow[r, phantom, "\circlearrowleft"] & X(i,j) \arrow[ur, "c^j_i"] \arrow[dr, "b"] \arrow[r, phantom, "\circlearrowleft"] & & \\
& \lim_J X(\hat{i},-) \arrow[ur, "a"] \arrow[r, "l^{\hat{i}}_j"'] & X(\hat{i},j) \arrow[u, "X(m,j)"'] & \operatorname{colim}_I X(-,\hat{j}) \arrow[uu, "\operatorname{colim}_I X(-,n)"'] &
\end{tikzcd}
$$

$\square$

Example 2. Let $f : X \times Y \to \mathbb{R}$ be a function. Then, provided they exist,

$$\sup_{x\in X} \inf_{y\in Y} f(x,y) \leq \inf_{y\in Y} \sup_{x\in X} f(x,y) \tag{7}$$

Proof. Consider the partially ordered set $\mathbb{R}$ as a category with morphisms $\leq$ and X, Y as discrete index categories. $\square$

Example 3. For abelian groups, there exist the morphism

$$\kappa : \bigoplus_{i\in\mathbb{N}} \prod_{j\in\mathbb{N}} \mathbb{Z} \to \prod_{j\in\mathbb{N}} \bigoplus_{i\in\mathbb{N}} \mathbb{Z}; \quad (a_{ij})_{i,j\in\mathbb{N}} \mapsto (a_{ji})_{i,j\in\mathbb{N}} \tag{8}$$

which is injective, but not surjective. For example, the element defined by $a_{ij} = \begin{cases} 1 & j \leq i \\ 0 & j > i \end{cases}$ is not in the image.

3 Filtered Limits - Finite Limits

Definition 1. A category $\mathcal{C}$ is called **filtered** if every finite diagram in $\mathcal{C}$ has a cocone.

Remark 1. Note that every category with a terminal object is filtered. This, thus, includes **Set**, **Top**, **Cat** and algebraic categories. However, for example $\bullet \leftarrow \bullet \rightarrow \bullet$ is not filtered.

Remark 2. A colimit (of shape I) in **Set** is explicitely given (dually to a limit) as a coequalizer of a coproduct. That is, the quotient of a disjoint union $\coprod_I X(i)$ under the equivalence relation which is generated by $x_i \sim x_{\hat{i}}$ if $\exists X(f) : X(i) \to X(\hat{i})$ with $X(f)(x_i) = x_{\hat{i}}$. (cf. a pushout of sets)

For a filtered index category I this amounts to $x_i \sim x_{\hat{i}}$ *if and only if* $\exists t \in I : X(f)(x_i) = X(g)(x_{\hat{i}})$ for some $f : i \to t$, $g : \hat{i} \to t$, because then there exists a cocone under that diagram and thus a morphism $X(i) \to X(\hat{i})$ making that diagram commute. Note that in the case of I not being filtered, this does not give an equivalence relation.

Theorem 4. *Let $X : I \times J \to \mathbf{Set}$ be diagram in the category of sets and let I be filtered and J finite (with n objects). Then*

$$\operatorname*{colim}_I \lim_J X \simeq \lim_J \operatorname*{colim}_I X \tag{9}$$

with the isomorphism being given by the canonical morphism described in theorem 3.

- More generally, this is true when replacing **Set** by any algebraic category. See Brandenburg [2016].
- With some subtleties, one can generalize this to index categories of higher cardinality. See Satz 5.2 in Gabriel und Ulmer [2006].
- In fact, a category I is filtered if and only if colimits of shape I commute with finite limits in **Set**. See section 2.13 of Borceux [1994].

Proof. The proof uses the explicit construction of limits in **Set** and of filtered colimits.

- κ is injective: Let $x, y \in \operatorname{colim}_I \lim_J X$ s.t. $\kappa(x) = \kappa(y)$. Then by the construction of the limit as subset of $\prod_J \operatorname{colim}_I X$ one may express this in coordinates as $(x_j)_J = \kappa(x) = \kappa(y) = (y_j)_J$ which means that $\forall j \in J : x_j = y_j$. However, by the explicit definition of the colimit over a filtered index category as $\coprod_I X(i,j)/\sim$ for fixed $j \in J$, this means that there exists a $t \in I$ s.t. for $X(i,j) \to_f X(t,j) \leftarrow_g X(\hat{i},j)$ we have $x \mapsto f(x) = g(y) \leftarrow\!\shortmid y$ for every $j \in J$. Applying the limit over J, this yields the same for $x \in \lim_J X(i,-)$ and $y \in \lim_J X(\hat{i},-)$ and thus by definition their equivalence classes are equal in $\operatorname{colim}_I \lim_J X$, showing injectivity.
- Let $(\overline{x}_j)_J \in \lim_J \operatorname{colim}_I X$ be arbitrary. Then for all $j \in J$ we have $\overline{x}_j \in \operatorname{colim}_I X(-,j)$, which is an equivalence class of elements in $\coprod_I X(i,j)$. Pick a representative $x_j \in X(i_j, j)$ which lies in $X(i_j, j)$ for some sufficient $i_j \in I$. Then we want to show that there exists a $t \in I$ s.t. there is a representative $x'_j \in X(t,j)$ for every $j \in J$, since then $x \in \lim_J X(t,-)$ and thus $\overline{x} \in \operatorname{colim}_I \lim_J X$ which gives the surjectivity. But this is precisely the case because J is finite.

$\square$

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