Kan Extensions

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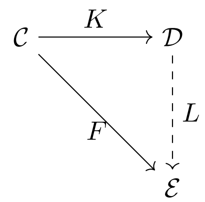
Example 1. (Extension of functor on poset) Consider the following situation: Let $(\mathbb{Z}, \leq)$ and $(\mathbb{R}, \leq)$ be the partially ordered set of integers and real numbers, respectively. Furthermore, consider a functor (i.e. a monotonically increasing function) $f : \mathbb{Z} \to \mathbb{R}$. Is it possible/how is it possible to extend this functor to $\mathbb{R}$? That is, does there exist a monotone function $g : \mathbb{R} \to \mathbb{R}$ s.t. $g|_{\mathbb{Z}} = f$ and what is that function explicitly?

1 Definition via universal property

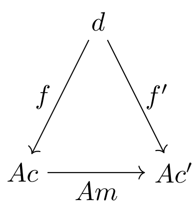
More generally, the previous example constitutes the following problem:

Let $F : \mathcal{C} \to \mathcal{E}$ and $K : \mathcal{C} \to \mathcal{D}$ be functors with a common domain. Does there exist an extension of F along K? That is, does there exist an $L : \mathcal{D} \to \mathcal{E}$ s.t. the following diagram commutes?



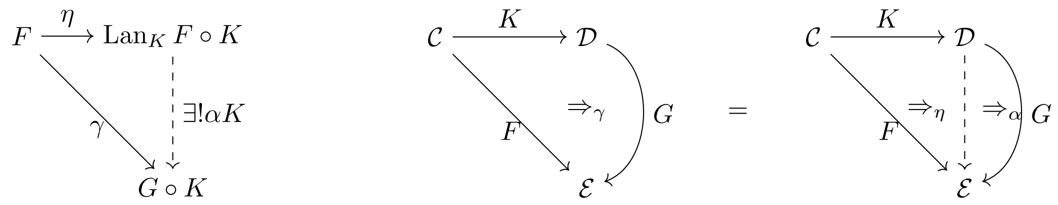
Such a functor may very well not exist. (Consider for example two morphisms $k \neq h \in \mathcal{C}$, but $K(h) = K(k) \in \mathcal{D}$ and $F(h) \neq F(k) \in \mathcal{E}$). And even if such an extension exists, by no means does it need to be unique.

Recall: Comma category Let $A : \mathcal{C} \to \mathcal{D}$ be a functor and let $d \in \mathcal{D}$. Then the category $d \downarrow A$ has as objects pairs $(c, f : d \to Ac)$ with $c \in \mathcal{C}$ and as morphisms $m : (c, f) \to (c', f')$ morphisms $m : c \to c'$ s.t. the following diagram commutes



Recall: Precomposition functor Let $K : \mathcal{C} \to \mathcal{D}$. Then there exists a functor $K^* : [\mathcal{D}, \mathcal{E}] \to [\mathcal{C}, \mathcal{E}]$ via $G \mapsto G \circ K$ and $\eta \mapsto \eta K$.

Definition 1 (via Universal Property)**.** Let $F : \mathcal{C} \to \mathcal{E}$ and $K : \mathcal{C} \to \mathcal{D}$ be functors with a common domain. Then (if it exists) the left Kan extension $(Lan_K F : \mathcal{D} \to \mathcal{E}, \eta : F \Rightarrow \mathrm{Lan}_K F \circ K)$ of F along K is the initial object in the category $F \downarrow K^*$. Explicitly this means that for any $(G : \mathcal{D} \to \mathcal{E}, \gamma : F \Rightarrow G \circ K)$ there exists a unique $\alpha : Lan_K F \Rightarrow G$ s.t. $\alpha K \circ \eta = \gamma$. In other words, the left diagram (in $[\mathcal{C}, \mathcal{E}]$) needs to commute or equivalently the two right diagrams need to be the equal:



Similarly the right Kan extension is the terminal object in $K^* \downarrow F$.

2 Characterization via Adjoints

The following proposition should make the interpretation of a Kan extension as a canonical pseudo extension more plausible (given one interprets adjunctions as pseudo inverses).

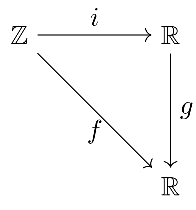
Theorem 1. *Let $K : \mathcal{C} \to \mathcal{D}$ be a functor and $\mathcal{E}$ a category. Then for any $F : \mathcal{C} \to \mathcal{E}$ the left (resp. right) Kan extension $\mathrm{Lan}_K(F)$ of F along K exists if and only if K^* is a right adjoint i.e. $L \dashv K^*$ for some functor $L : [\mathcal{C}, \mathcal{E}] \to [\mathcal{D}, \mathcal{E}]$ (resp. K^* is a left adjoint). If that is the case, then $\mathrm{Lan}_K \dashv K^*$ (resp. $K^* \dashv \mathrm{Ran}_K$).*

Proof. We have the following equivalent statements

- K^* is a right adjoint i.e. $L \dashv K^*$ for some functor $L : [\mathcal{C}, \mathcal{E}] \to [\mathcal{D}, \mathcal{E}]$
- $\forall F \in [\mathcal{C}, \mathcal{E}]$ the functor $\mathrm{Hom}_{[\mathcal{C},\mathcal{E}]}(F, K^*(-))$ is representable (by $L(F)$)
- $\forall F \in [\mathcal{C}, \mathcal{E}]$ the category $F \downarrow K^* = \int \mathrm{Hom}_{[\mathcal{C},\mathcal{E}]}(F, K^*(-))$ has an initial object (which is $L(F)$)
- $\forall F \in [\mathcal{C}, \mathcal{E}]$ the left Kan extension $\mathrm{Lan}_K(F)$ exists (which then coincides with $L(F)$) i.e. $\mathrm{Lan}_K = L$

□

Example 2. (Continuation of example 1) Consider again the situation

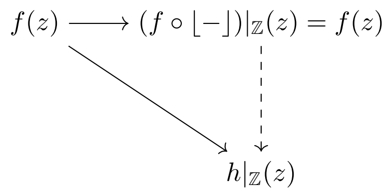


The inclusion induces the functor $i^* : [\mathbb{R}, \mathbb{R}] \to [\mathbb{Z}, \mathbb{R}], f \mapsto f \circ i$ from the set of monotone real valued functions on $\mathbb{R}$ to those on $\mathbb{Z}$. In order to show that a left Kan extension exists, we are looking for functor: $L : [\mathbb{Z}, \mathbb{R}] \to [\mathbb{R}, \mathbb{R}]$ s.t. $L \dashv i^*$ i.e. $\forall g \in [\mathbb{R}, \mathbb{R}], f \in [\mathbb{Z}, \mathbb{R}]$ we have $\hom(L(f), g) \simeq \hom(f, \underbrace{i^*(g)}_{g|_{\mathbb{Z}}})$. Since the arguments of the hom sets are functors, the morphisms are natural transformation, which means that $L(f)$ is such that

$$\forall x \in \mathbb{R} : L(f)(x) \leq g(x) \Leftrightarrow \forall z \in \mathbb{Z} : f(x) \leq g|_{\mathbb{Z}}(x) \tag{1}$$

Defining L by precomposing g with the floor function satisfies equation (1). So the left adjoint $L \dashv i^*$ is given by $L(f)(x) = f(\lfloor x \rfloor)$.

In this context, the left Kan extension of a fixed $f : \mathbb{Z} \to \mathbb{R}$ is then given by a monotonically increasing function $\mathrm{Lan}_i(f) : \mathbb{R} \to \mathbb{R}, x \mapsto f(\lfloor x \rfloor)$ s.t. $f(z) \leq (f \circ \lfloor - \rfloor)|_{\mathbb{Z}}(z) = f(z)$ s.t. for any other monotone function $h : \mathbb{R} \to \mathbb{R}$ with $f(z) \leq h|_{\mathbb{Z}}(z)$ we have that $f(\lfloor x \rfloor) \leq h(x)$ s.t. $\eta \circ \alpha = \gamma$. As a diagram in $[\mathbb{Z}, \mathbb{R}]$ this is



where an arrow $a \to b$ exists iff $a \leq b$.

3 Explicit Description of Kan extensions

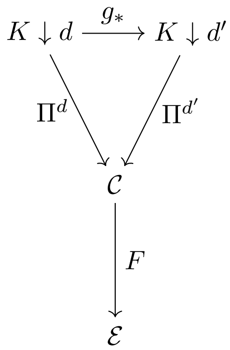
Theorem 2 (Explicit construction of Kan extensions as limits. [Riehl, 2017])**.** *Let $F : \mathcal{C} \to \mathcal{E}$ and $K : \mathcal{C} \to \mathcal{D}$ be functors with a common domain. If for every $d \in \mathcal{D}$ the colimit*

$$\mathrm{colim}(K \downarrow d \xrightarrow{\Pi^d} \mathcal{C} \xrightarrow{F} \mathcal{E}) \tag{2}$$

exists, then $\mathrm{Lan}_K(F)(d) = \mathrm{colim}(K \downarrow d \to \mathcal{C} \to \mathcal{E})$ on objects defines the left Kan extension. The natural transformation $\eta : F \Rightarrow \mathrm{Lan}_K F \circ K$ is given by the colimit cones given by the $d \in \mathcal{D}$. Similarly, the right Kan extension is given by $\mathrm{Ran}_K F(d) = \lim(d \downarrow K \xrightarrow{\Pi_d} \mathcal{C} \xrightarrow{F} \mathcal{E})$ for every $d \in \mathcal{D}$ and the natural transformation is given by the limit cones.

Proof. Firstly, we construct the functor $\mathrm{Lan}_K(F)$ and the natural transformation $\eta : F \Rightarrow \mathrm{Lan}_K F \circ K$. Then we will show that these have the desired universal property.

Constructing $(\mathrm{Lan}_K(F), \eta)$**:** On objects, we have $\mathrm{Lan}_K(F)(d) = \mathrm{colim}(K \downarrow d \to \mathcal{C} \to \mathcal{E})$. For morphisms, consider for a fixed morphism $g : d \to d'$ the induced functor $g_* : K \downarrow d \to K \downarrow d'$ via $(c, f : Kc \to d) \mapsto (c, g \circ f : Kc \to d \to d')$ and identity on morphisms. Then the following diagram commutes.

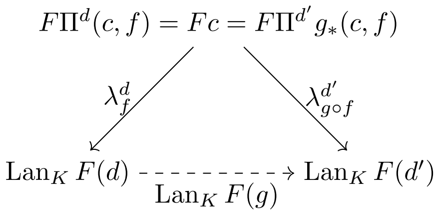


where Π^d is the forgetful functor $K \downarrow d \to \mathcal{C}$ via $(c, f) \mapsto c$.

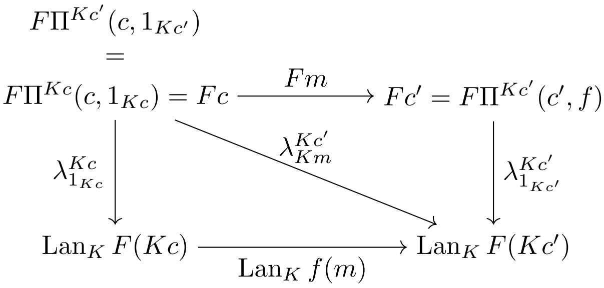
For a fixed $F : \mathcal{C} \to \mathcal{E}$ consider the diagram $F \circ \Pi^{d'}$ and its $(\operatorname{Lan} F(d'), \lambda^{d'})$. That diagram induces another diagram $F \circ \Pi^{d'} \circ g_* = F \circ \Pi^d$ and a co-cone $(\operatorname{Lan} F(d'), \lambda^{d'} g_*)$over it. Explicitly, for the component based at $(c, f) \in K \downarrow d$ we have

$$(\lambda^{d'} g_*)_f : F \circ \Pi^{d'} \circ g_*(c, f) = F \circ \Pi^{d'}(c, g \circ f) \to \operatorname{Lan}_K F(d') \tag{3}$$

However, now by the universal property of $\operatorname{Lan}_K F(d)$ as the colimit of $F \circ \Pi^d$ there exists a unique morphism $\operatorname{Lan}_K F(d) \to \operatorname{Lan}_K F(d')$ making the following commute.



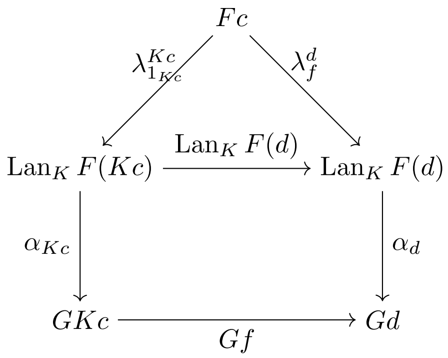
The components of the natural transformation $\eta : F \Rightarrow \operatorname{Lan}_K F \circ K$ is given by $\lambda^{Kc}_{1_{Kc}}$. To prove naturality consider $m : c \to c'$ and the naturality square



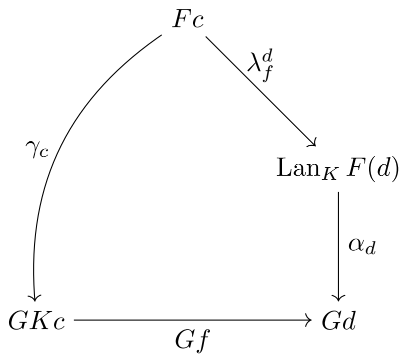
The lower left triangle commutes by what was shown before and the upper left since $\lambda^{Kc'}$ is a cone.

Existence & Uniqueness of α: It is left to show that $(\mathrm{Lan}_K(F), \eta)$ fulfils the universal property given in definition 1. So let $(G : \mathcal{D} \to \mathcal{E}, \gamma : F \Rightarrow GK)$ be arbitrary. We want to define a unique $\alpha : \mathrm{Lan}_K(F) \Rightarrow G$ s.t. $\gamma = \alpha K \circ \eta$.

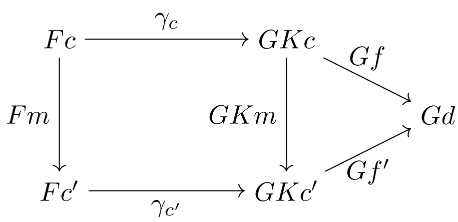
For each $d \in \mathcal{D}$ the naturality of α implies that (in particular) the lower square in the follow diagram needs to commute.



By the construction of $\mathrm{Lan}_K F$ this means that the outer pentagon commutes as well. Thus with the factorization requirement $\forall c \in \mathcal{C} : \alpha_{Kc} \circ \underbrace{\eta_c}_{=\lambda^{Kc}_{1_{Kc}}} = \gamma_c$ we ultimately have that α_d needs to be s.t.

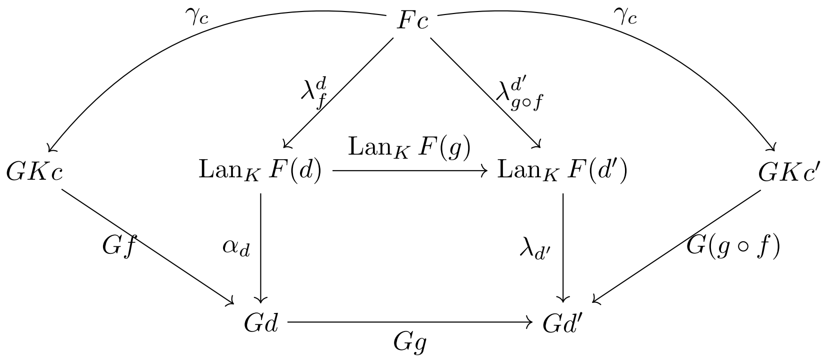


commutes. Now $Gf \circ \gamma_c$ defines a co-cone (see diagram below) over the diagram $F\Pi^d$ indexed by $f \in K \downarrow d$ with nadir Gd, there exists a unique α_d making the diagram commute. Hence for every γ, the desired α exists and is unique.
Consider the following diagram to see that $Gf \circ \gamma_c$ is a co-cone.



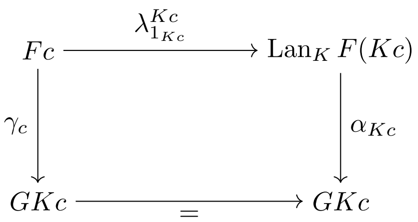
Here the left square commutes since γ is natural and the right triangle commutes since G is a functor.

Naturality of α: Consider the naturality square in the middle



The upper triangle commutes, so it suffices to show that the inner pentagon commutes. This is indeed the case, since the outer pentagon commutes by definition and each outer square commutes because of the definition of α_d.

Factorization via α: Finally, for the factorization we want to show that $\forall c \in \mathcal{C}$ we have $\alpha_{Kc} \circ \eta_c = \gamma_c$ i.e. we want to show that the following diagram commutes



This is a special case of the defining diagram of α_d by considering $c \in \mathcal{C}$ s.t. $d = Kc$.

□

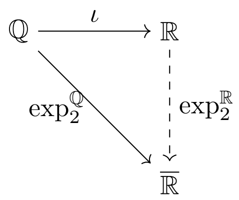
Corollary 1. *[Riehl, 2017] Let $\mathcal{C}$ be small, $\mathcal{D}$ locally small and $K : \mathcal{C} \to \mathcal{D}$ a functor and $\mathcal{E}$ a category. Then*

(i) If $\mathcal{E}$ is co-complete, then $\mathrm{Lan}_K \dashv K^*$ exists and is given by the formula in theorem 2.

(ii) If $\mathcal{E}$ is complete, then $K^* \dashv \mathrm{Ran}_K$ exists and is given by the formula in theorem 2.

Proof. If $\mathcal{C}$ is small and $\mathcal{D}$ is locally small, then $K \downarrow d$ and $d \downarrow K$ are small. Thus the co-completeness/completeness together with the theorem give the result. □

Example 3. Consider the exponentiation map with base 2 defined on rational numbers. We want to extend this function to the real numbers. Firstly, note that $q \mapsto 2^q$ is a monotone map $\mathbb{Q} \to \overline{\mathbb{R}}$. So we have the following situation



where $\exp_2^{\mathbb{R}}$ is the supposed extension.

The category $\overline{\mathbb{R}}$ is complete and co-complete and $\mathbb{Q}$ and $\mathbb{R}$ are small, so by corollary 1 the left and right Kan extensions exist. With theorem 2 they admit an explicit description as a colimit, which in the case of a partially ordered set is the supremum indexed by the category $\iota \downarrow x$. The object of this category are such $q \in \mathbb{Q}$ s.t. $q \leq x$. Here this means that for any $x \in \mathbb{R}$ we have

$$2^x := (\mathrm{Lan}_i \exp_2)(x) = \sup_{p\in\mathbb{Q}, p\leq x} 2^p \tag{4}$$

The right Kan extension, (via a limit construction indexed by the category $(x \downarrow \iota)$ giving $(\mathrm{Ran}_i \exp_2)(x) = \inf_{p\in\mathbb{Q}, x\leq p} 2^p$. In this (special) case they coincide.

Example 4. (Induction of Group-Representations) Consider a subgroup $H \subseteq G$ and a group representation $\rho : \mathbf{B}H \to \mathbf{Vec}_k$[1]. We want to give a canonical representation of G which is induced by ρ. The categories $\mathbf{B}H$ and $\mathbf{B}G$ are small and $\mathbf{Vec}_k$ is complete and co-complete. Thus by corollary 1 the left and right Kan extensions exist. They are then referred to as induction and co-induction.

References

[Brandenburg 2016] Brandenburg, Martin: *Einführung in die Kategorientheorie: Mit ausführlichen Erklärungen und zahlreichen Beispielen.* Springer-Verlag, 2016

[Riehl 2017] Riehl, Emily: *Category theory in context.* Courier Dover Publications, 2017

[1] Here $\mathbf{B}A$ denotes the groupoid associated to the group H